108 年題庫班

講義勘誤檔案

第 1 本 Laplace Fourier

喻超凡博士編

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1.2 精選歷試

《 **題型 1** 》基本性質

以下關於 Laplace transform (記爲：$\mathscr{L}(\ \cdot\ )$) 的敘述，何者爲錯誤。

(A) 滿足 $\mathscr{L}(f)+\mathscr{L}(g)=\mathscr{L}(f+g)$。

(B) 其逆轉換 $\mathscr{L}^{-1}(\ \cdot\ )$ 亦滿足 $\mathscr{L}^{-1}(f)+\mathscr{L}^{-1}(g)=\mathscr{L}^{-1}(f+g)$。

(C) 若 f, g 爲連續函數，且 $\mathscr{L}(f)=\mathscr{L}(g)$, 則 $f=g$。

(D) 以上 (a) (b) (c) 敘述皆正確。

(E) 以上 (a) (b) (c) (d) 敘述皆錯誤。 《**喻超凡、喻超弘** 108中山電機乙》

《**解**》

(A) False：要求 $\mathscr{L}(f)$、$\mathscr{L}(g)$ 要存在，才成立，反例 $f(t)=t+e^{t^2}$、$g(t)=t-e^{t^2}$，故 $f(t)+g(t)=2t$，$\mathscr{L}(f)$、$\mathscr{L}(g)$ 均不存在，但 $\mathscr{L}(f+g)=\dfrac{2}{s^2}$，故

$$\mathscr{L}(f)+\mathscr{L}(g)\neq\mathscr{L}(f+g)$$

(B) False：同 (A) 的性質，要求 $\mathscr{L}^{-1}(f)$、$\mathscr{L}^{-1}(g)$ 要存在，才成立。

(C) True：爲 Mathias Lerch's 定理。

故選 (A)、(B)、(D)

《 **題型 2** 》基本性質

以下關於 Laplace transform (記爲: $\mathscr{L}(\ \cdot\ )$) 的敘述，何者爲正確。

(A) $\mathscr{L}(t^3)=\dfrac{1}{s^3}$。

(B) 若 $a>0$ 且 $\mathscr{L}^{-1}(F(s))=f(t)$，則 $\mathscr{L}^{-1}(e^{-as}F(s))=f(t-a)u(t-a)$，其中 $u(t)$ 滿足 $u(t)=1, \forall t\geq 0, u(t)=0$，$\forall t<0$。

(C) 若 $f(0)=0$，則 $\mathscr{L}(\ddot{f})=s^2\mathscr{L}(f)$。

(D) $\mathscr{L}^{-1}(F(s-a))=e^a\mathscr{L}^{-1}(F(s))$。

(E) 以上 (a) (b) (c) (d) 敘述皆正確。 《**喻超凡、喻超弘** 108中山電機乙》

《解》

(A) False：$\mathscr{L}(t^3) = \dfrac{3!}{s^4}$

(B) True：定理。

(C) False：$\mathscr{L}(\ddot{f}) = s^2\mathscr{L}(f) - sf(0) - f'(0) = s^2\mathscr{L}(f) - f'(0)$

(D) False：$\mathscr{L}^{-1}(F(s-a)) = e^{at}\mathscr{L}^{-1}(F(s))$

選 (B)

《題型 3》基本性質

The Laplace transform of a function $f(t)$, for all $t \geq 0$, is defined by the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

The Laplace transform, however, does not exist for all functions.

(a) State the conditions for the Laplace transform of the function $f(t)$ to exist. (5%)

(b) Test for the existence of the Laplace transform for the following functions. If the Laplace transform for the given function exists, evaluate it. (10%)

(1) $e^{2t}t^{-\frac{1}{2}}$ (2) e^{t^2} 《喻超凡、喻超弘 107 交大機械丙》

《解》

(a) Laplace 轉換存在之充分條件：設 $f(t)$ 滿足

(1) $f(t)$ 在 $t \geq 0$ 時為片段連續。

(2) $f(t)$ 在 $t \to \infty$ 為指數階函數。

(b)

(1) 因 $\lim\limits_{t\to 0} e^{2t}t^{-\frac{1}{2}}$ 不存在，故不滿足存在定理，因

$$\mathscr{L}\{t^{-\frac{1}{2}}\} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$$

故

$$\mathscr{L}\{e^{2t}t^{-\frac{1}{2}}\} = \frac{\sqrt{\pi}}{(s-2)^{\frac{1}{2}}}$$

$$= \frac{2}{s+2}(\frac{1}{s^2} - \frac{3s+1}{s^2}e^{-3s})$$
$$= \frac{1}{s^2} - \frac{1}{2s} + \frac{1}{2(s+2)} - \{\frac{1}{s^2} + \frac{5}{2s} - \frac{5}{2(s+2)}\}e^{-3s}$$

故

$$h(t) = t - \frac{1}{2} + \frac{1}{2}e^{-2t} - \{(t-3) + \frac{5}{2} - \frac{5}{2}e^{-2(t-3)}\}H(t-3)$$

《 **題型 13** 》L–T

(20%) Find the Laplace transform (and show the details of your work) for the following functions:

(a) $t\,e^{-t}\ \cosh(2t)$ (10%)

(b) $\sinh(at)\ \sin(at)$ (10%)

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《解》

(a) 因 $\mathscr{L}\{\cosh(2t)\} = \frac{s}{s^2-4}$，故 $\mathscr{L}\{e^{-t}\cosh(2t)\} = \frac{s+1}{(s+1)^2-4}$，因此

$$\mathscr{L}\{t\,e^{-t}\ \cosh(2t)\} = -\frac{d}{ds}(\frac{s+1}{(s+1)^2-4}) = \frac{s^2+2s+5}{\{(s+1)^2-4\}^2}$$

(b)

$$\begin{aligned}\mathscr{L}\{\sinh(at)\cdot\sin(at)\} &= \mathscr{L}\{\frac{e^{at}-e^{-at}}{2}\cdot\sin(at)\} \\ &= \frac{1}{2}\mathscr{L}\{e^{at}\cdot\sin(at) - e^{-at}\cdot\sin(at)\} \\ &= \frac{1}{2}\{\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\} \\ &= \frac{2a^2s}{s^4+4a^4}\end{aligned}$$

《 **題型 14** 》L–T

(10%) Find the Laplace transform of following functions

$$te^{-t}\cos t + t^2\sin t$$

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《解》

$$\mathscr{L}\{te^{-t}\cos t\} = -\frac{d}{ds}\{\frac{s+1}{(s+1)^2+1}\} = \frac{s(s+2)}{(s^2+2s+2)^2}$$

$$\mathscr{L}\{t^2\sin t\} = (-1)^2\frac{d^2}{ds^2}(\frac{1}{s^2+1}) = \frac{d}{ds}(-\frac{2s}{(s^2+1)^2}) = \frac{6s^2-2}{(s^2+1)^3}$$

故

$$\mathscr{L}\{te^{-t}\cos t + t^2\sin t\} = -\frac{s(s+2)}{(s^2+2s+2)^2} + \frac{6s^2-2}{(s^2+1)^3}$$

《 題型 15 》L–T

Place fine y if (1) $\mathscr{L}\{y\} = \dfrac{s+1}{s^2+s-6}$ (2) $\mathscr{L} = \dfrac{1}{s(s^2+4)}$ (16%)

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《解》 (1) $y = \mathscr{L}^{-1}\{\dfrac{s+1}{s^2+s-6}\} = \mathscr{L}^{-1}\{\dfrac{3}{5(s-2)} + \dfrac{2}{5(s+3)}\} = \dfrac{3}{5}e^{2t} + \dfrac{2}{5}e^{-3t}$

(2) $y = \mathscr{L}^{-1}\{\dfrac{1}{s(s^2+4)}\} = \displaystyle\int_0^t \frac{1}{2}\sin 2t\,dt = \frac{1}{4}(1-\cos 2t)$

《 題型 16 》L–T 《喻超凡、喻超弘 107中興電機》

(a) (10%) Determine the Laplace transform for the function g defined by

$$g(t) = \begin{cases} 0 & \text{, if } 0 \le t < 2 \\ t^2+1 & \text{, if } t \ge 2 \end{cases}$$

(b) (10%) Solve the following initial value problem by the Laplace transform:

$$y'' + 4y' + 3y = e^t\ ;\ y(0) = 0\,,\ y'(0) = 2$$

《解》

(a) 因

$$g(t) = \begin{cases} 0 & \text{, if } 0 \le t < 2 \\ t^2+1 & \text{, if } t \ge 2 \end{cases} = (t^2+1)H(t-2)$$

故

$$\begin{aligned}\mathscr{L}\{g(t)\} &= \mathscr{L}\{(t^2+1)H(t-2)\} = e^{-2s}\mathscr{L}\{(t+2)^2+1\} \\ &= e^{-2s}\mathscr{L}\{t^2+4t+5\} = e^{-2s}(\frac{2}{s^3}+\frac{4}{s^2}+\frac{5}{s})\end{aligned}$$

(b) 對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - y'(0) + 4\{sY(s) - y(0)\} + 3Y(s) = \frac{1}{s-1}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$，整理可得

$$(s^2+4s+3)Y(s) = 2 + \frac{1}{s-1} = \frac{2s-1}{s-1}$$

故

$$Y(s) = \frac{2s-1}{(s+1)(s+3)(s-1)} = \frac{3}{4(s+1)} - \frac{7}{8(s+3)} + \frac{1}{8(s-1)}$$

因此

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t} + \frac{1}{8}e^{t}$$

《 題型 17 》L–T

(1) (10%) Find the Laplace transform of $f(t) = (\sin at)^2$

(2) (10%) Find the inverse of Laplace transform of $F(s) = \dfrac{e^{-s}}{s^2+3s+2}$

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《解》

(1)

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{\sin^2 at\} = \mathscr{L}\{\frac{1-\cos 2at}{2}\} = \frac{1}{2s} - \frac{1}{2}\cdot\frac{s}{s^2+4a^2}$$

(2) 因

$$\begin{aligned}\mathscr{L}^{-1}\{\frac{1}{s^2+3s+2}\} &= \mathscr{L}^{-1}\{\frac{1}{(s+2)(s+1)}\}\\ &= \mathscr{L}^{-1}\{\frac{1}{s+1} - \frac{1}{s+2}\}\\ &= e^{-t} - e^{-2t}\end{aligned}$$

故

$$\mathscr{L}^{-1}\{F(s)\} = \frac{e^{-s}}{s^2+3s+2} = \{e^{-(t-1)} - e^{-2(t-1)}\}H(t-1)$$

《 題型 18 》L–T

(5%) Find the Laplace transform for $f(t) = e^{2t}\cos t$.

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$$= -\frac{1}{t}\mathscr{L}^{-1}\{\frac{1}{1+(\frac{w}{s})^2}\cdot(-\frac{w}{s^2})\} = \frac{1}{t}\mathscr{L}^{-1}\{\frac{w}{s^2+w^2}\}$$

$$= \frac{\sin wt}{t}$$

(b)

$$\mathscr{L}^{-1}\{\ln\frac{s+a}{s+b}\} = -\frac{1}{t}\mathscr{L}^{-1}\{\frac{d}{ds}\ln\frac{s+a}{s+b}\} = -\frac{1}{t}\mathscr{L}^{-1}\{\frac{1}{s+a}-\frac{1}{s+b}\}$$

$$= -\frac{1}{t}(e^{-at}-e^{-bt})$$

《 題型 40 》Laplace 反轉換

(10%) Find the Inverse Laplace Transform of $\dfrac{4(e^{-2s}-2e^{-5s})}{s^2+4}$

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《解》 令

$$F(s) = \frac{4(e^{-2s}-2e^{-5s})}{s^2+4}$$

則兩端取 Laplace 反轉換可得

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\{\frac{4e^{-2s}}{s^2+4}-\frac{8e^{-5s}}{s^2+4}\} \\ &= 2\sin[2(t-2)]H(t-2) - 4\sin[2(t-5)]H(t-5) \end{aligned}$$

《 題型 41 》Laplace 反轉換

(15%) Given $F(s) = \mathscr{L}(f)$, find $f(t)$. Show the details of your work. (L is constant)

(a) $\dfrac{s}{L^2s^2+1/4\pi^2}$ (b) $\dfrac{2}{s^4}-\dfrac{48}{s^6}$ (c) $\dfrac{90}{(s+\sqrt{3})^6}$

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《解》

(a)

$$f(t) = \mathscr{L}^{-1}\{\frac{s}{L^2s^2+1/4\pi^2}\} = \frac{1}{L^2}\mathscr{L}^{-1}\{\frac{s}{s^2+\frac{1}{4\pi^2L^2}}\} = \frac{1}{L^2}\cos(\frac{t}{2\pi L})$$

故

$$\mathscr{L}\{g(t)\} = \mathscr{L}\{3\cos t\, H(t-\pi)\} = 3e^{-\pi s}\mathscr{L}\{\cos(t+\pi)\} = -\frac{3s}{s^2+1}e^{-\pi s}$$

對 ODE 兩端取 L–T，可得

$$sY(s) - y(0) + Y(s) = \mathscr{L}\{g(t)\}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$，整理可得

$$(s+1)Y(s) = 5 - \frac{3s}{s^2+1}e^{-\pi s}$$

即

$$Y(s) = \frac{5}{s+1} - \frac{3s}{(s^2+1)(s+1)}e^{-\pi s} = \frac{5}{s+1} - \{-\frac{3}{2(s+1)} + \frac{3(s+1)}{2(s^2+1)}\}e^{-\pi s}$$

故

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\ &= 5e^{-t} - \{-\frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\cos(t-\pi) + \frac{3}{2}\sin(t-\pi)\}H(t-\pi) \end{aligned}$$

《 **題型 54** 》解常係數 ODE

(5%) Use Laplace Transform method to solve the ODE

$$y'' + y = \delta(t-\pi)\ ;\ y(0) = 0\,,\ y'(0) = 0$$

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《**解**》 對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = e^{-\pi s}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$，整理可得

$$Y(s) = \frac{1}{s^2+1}e^{-\pi s}$$

故

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \sin(t-\pi)H(t-\pi)$$

《解》對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - y'(0) + 5\{sY(s) - y(0)\} + 6Y(s) = \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$，整理可得

$$(s^2 + 5s + 6)Y(s) = s + 5 + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}$$

故

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+3)(s+2)} + \frac{1}{s(s+3)(s+2)}(e^{-s} - e^{-2s}) \\ &= \frac{3}{s+2} - \frac{2}{s+3} + \{\frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}\}e^{-s} \\ &\quad -\{\frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)}\}e^{-2s} \end{aligned}$$

故

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\ &= 3e^{-2t} - 2e^{-3t} + \{\frac{1}{6} - \frac{1}{2}e^{-2(t-1)} + \frac{1}{3}e^{-3(t-1)}\}H(t-1) \\ &\quad -\{\frac{1}{6} - \frac{1}{2}e^{-2(t-2)} + \frac{1}{3}e^{-3(t-2)}\}H(t-2) \end{aligned}$$

《**題型 59**》解常係數 ODE

Use the Laplace transform to solve the problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = f(t)$$

where $f(t) = \begin{cases} 0 & ,\ 0 \le t < 1 \\ 2 & ,\ t \ge 1 \end{cases}$, $x(0) = 1$, and $x'(0) = -1$. You may express $f(t)$ in term of a unit step function. (10%) 《喻超凡、喻超弘 108 清大工科》

《解》因

$$f(t) = \begin{cases} 0 & ,\ 0 \le t < 1 \\ 2 & ,\ t \ge 1 \end{cases} = 2H(t-1)$$

故

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{2H(t-1)\} = \frac{2}{s}e^{-s}$$

故

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\ &= \mathscr{L}^{-1}\{\frac{s}{s^2+2}+(\frac{1}{2s}-\frac{s}{2(s^2+2)})(e^{-\frac{\pi}{\sqrt{2}}s}-e^{-\sqrt{2}\pi s})\} \\ &= \cos(\sqrt{2}t)+\{\frac{1}{2}-\frac{1}{2}\cos\sqrt{2}(t-\frac{\pi}{\sqrt{2}})\}u(t-\frac{\pi}{\sqrt{2}}) \\ &\quad -\{\frac{1}{2}-\frac{1}{2}\cos\sqrt{2}(t-\sqrt{2}\pi)\}u(t-\sqrt{2}\pi) \end{aligned}$$

《 **題型 61** 》解常係數 ODE

Use Laplace transform tor solve the following equation :

$$y''+5y'+6y=1-u(t-1)\ ,\ y(0)=0\ ,\ y'(0)=0$$

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《**解**》 對 ODE 兩端取 L–T , 可得

$$s^2Y(s)-sy(0)-y'(0)+5\{sY(s)-y(0)\}+6Y(s)=\frac{1}{s}-e^{-s}\frac{1}{s}$$

整理可得

$$(s^2+5s+6)Y(s)=\frac{1}{s}-\frac{1}{s}e^{-s}$$

故

$$\begin{aligned} Y(s) &= \frac{1}{s(s+2)(s+3}-\frac{1}{s(s+2)(s+3)}e^{-s} \\ &= \frac{1}{6s}-\frac{1}{2(s+2)}+\frac{1}{3(s+3)}-\{\frac{1}{6s}-\frac{1}{2(s+2)}+\frac{1}{3(s+3)}\}e^{-s} \end{aligned}$$

因此

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\frac{1}{6}-\frac{1}{2}e^{-2t}+\frac{1}{3}e^{-3t}-\{\frac{1}{6}-\frac{1}{2}e^{-2(t-1)}+\frac{1}{3}e^{-3(t-1)}\}\,u(t-1)$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$、$\mathscr{L}\{f(t)\} = F(s)$，整理可得

$$(s^2+2s+2)Y(s) = F(s)$$

故

$$Y(s) = \frac{1}{s^2+2s+2}\cdot F(s)$$

故

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\{\frac{1}{s^2+2s+2}\cdot F(s)\} = \mathscr{L}^{-1}\{\frac{1}{s^2+2s+2}\} * \mathscr{L}^{-1}\{F(s)\} \\ &= (e^{-t}\sin t) * f(t) = \int_0^t \{e^{-(t-\tau)}\sin(t-\tau)\}f(\tau)\,d\tau \end{aligned}$$

《 **題型 86** 》解常係數 ODE

Pleases solve the following equation：(5%)

$\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = \delta(t-\pi) + \delta(t-3\pi)$ with $y(0) = 1$, $y'(0) = 1$.

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《**解**》

(D) 題目有誤，應該改為

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = \delta(t-\pi) + \delta(t-3\pi)$$

對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - y'(0) + 4\{sY(s) - y(0)\} + 13Y(s) = e^{-\pi s} + e^{-3\pi s}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$，整理可得

$$(s^2+4s+13)Y(s) = s+5+e^{-\pi s} - e^{-3\pi s}$$

故

$$Y(s) = \frac{s+5}{(s+2)^2+3^2} + \frac{1}{(s+2)^2+3^2}(e^{-\pi s} + e^{-3\pi s})$$

則

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\ &= e^{-2t}\cos 3t + e^{-2t}\sin 3t \\ &\quad + \frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)H(t-\pi) + \frac{1}{3}e^{-2(t-3\pi)}\sin 3(t-3\pi)H(t-3\pi) \end{aligned}$$

即

$$\begin{cases} (s^2+s)X(s)+sY(s)=s \\ -2sX(s)+(s^2-s)Y(s)=-s \end{cases}$$

可解得

$$X(s)=\frac{s}{s^2+1}\ ,\ Y(s)=\frac{1-s}{s^2+1}$$

故

$$x(t)=\mathscr{L}^{-1}\{X(s)\}=\cos t\ ,\ y(t)=\mathscr{L}^{-1}\{Y(s)\}=\sin t-\cos t$$

《**題型 157**》L–T 解 PDE

(15%) The wave equation :

$$c^2u_{xx}-u_{tt}=0\ ,\ 0<x<\infty\ ;\ 0<t<\infty$$

with initial conditions : $u(x\,,\,0)=u_t(x\,,\,0)=0\ ;\ 0<x<\infty$
and boundary conditions : $u(0\,,\,t)=f(t)\ ,\ |u(\infty\,,\,t)|<\infty\ ;\ 0<t<\infty$
Solve for $u(x\,,\,t)$ (DO NOT use separation of variables).

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《**解**》對 PDE 兩端 t 變數取 L–T , 可得

$$c^2\frac{d^2U}{dx^2}-\{s^2U(x\,,\,s)-s\,u(x\,,\,0)-u_t(x\,,\,0)\}=0$$

整理可得

$$\frac{d^2U}{dx^2}-\frac{s^2}{c^2}U(x\,,\,s)=0$$

故

$$U(x\,,\,s)=c_1e^{\frac{s}{c}x}+c_2e^{-\frac{s}{c}x}$$

由 $\lim\limits_{x\to\infty}|u(x\,,\,t)|<\infty$, 即 $\lim\limits_{x\to\infty}|U(x\,,\,s)|<\infty$, 則 $c_1=0$, 再由

$$U(0\,,\,s)=\mathscr{L}\{u(0\,,\,t)\}=\mathscr{L}\{f(t)\}=F(s)=c_2$$

故

$$U(x\,,\,s)=F(s)\,e^{-\frac{x}{c}s}$$

因此

$$u(x\,,\,t)=\mathscr{L}^{-1}\{U(x\,,\,s)\}=\mathscr{L}^{-1}\{F(s)\,e^{-\frac{x}{c}s}\}=f(t-\frac{x}{c})\,H(t-\frac{x}{c})$$

(a) $t \geq 0$, 且 $c = 0$

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds = \sum_{c\text{ 的左邊}} \text{Res}\{F(s)e^{st}\} = \text{Res}f(-1) = -\frac{1}{2}e^{-t}$$

$t < 0$, 且 $c = 0$

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds = -\sum_{c\text{ 的右邊}} \text{Res}\{F(s)e^{st}\} = -\text{Res}f(1) = -\frac{1}{2}e^{t}$$

(b) $c > 1$, 則

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds = \begin{cases} 0 & t < 0 \\ \frac{1}{2}(e^t - e^{-t}) & t \geq 0 \end{cases}$$

《 **題型 162** 》雙邊 L–T 《**喻超凡、喻超弘** 94中興電機乙》

(15%) Given a time function $f(t)$ for $-\infty < t < \infty$, its two-sided (bilateral) Laplace transform is defined by $F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$. However, $F_b(s)$ does not always exit. If there exists a real positive number R and, for some two real numbers α and β, the time function $f(t)$ satisfies $f(t) = \begin{cases} Re^{\alpha t} & t > 0 \\ Re^{\beta t} & t < 0 \end{cases}$, then $F_b(s)$ converges for the region of convergence $\alpha < Re(s) < \beta$ where $Re(s)$ denotes the real part of the complex variable s. With this definition, answer the following two questions:

(a) Given $f(t) = e^{-a|t|}$, $a > 0$, $-\infty < t < \infty$, find its two-sided Laplace transform (5%) and region of convergence. (5%)

(b) Given $F_b(s) = \dfrac{2s+3}{(s+1)(s+2)}$ and the region of convergence $-2 < Re(s) < -1$, find its inverse Laplace transform $f(t)$. (5%)

《提示》 本題詳細內容可參閱 ”信號與系統” 的相關書籍 。

《解》

(a)

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \mathscr{L}\{e^{-a|t|}\} \\ &= \int_0^{\infty} e^{-at}e^{-st}dt + \int_{-\infty}^{0} e^{at}e^{-st}\,dt \\ &= -\frac{e^{-(s+a)t}}{s+a}\bigg|_0^{\infty} + \frac{e^{(a-s)t}}{a-s}\bigg|_{-\infty}^{0} \\ &= \frac{1}{s+a} + \frac{1}{a-s} = \frac{-2a}{s^2-a^2} \end{aligned}$$

(iii) 半波整流：

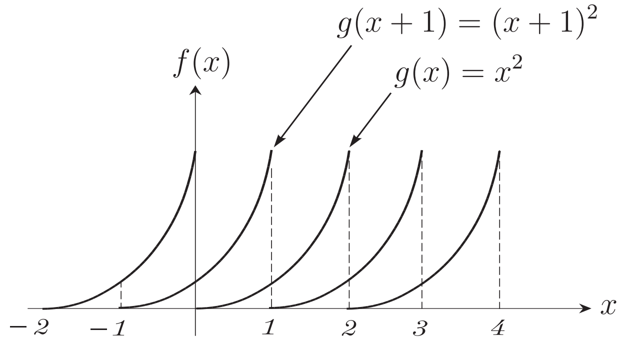
$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \sin(x-n\pi)\cdot H(x-n\pi) \\
&= \sin x \cdot H(x) + \sin(x-\pi)\cdot H(x-\pi) \\
&\quad + \sin(x-2\pi)\cdot H(x-2\pi) + \cdots
\end{aligned}
$$

故 $f(x) = \begin{cases} \sin x & ; 0 \le x < \pi \\ 0 & ; \pi \le x \le 2\pi \end{cases}$; $f(x) = f(x+2\pi)$, $x \ge 0$

(b) 函數有重疊型：例如：設 $g(x) = x^2\{H(x) - H(x-2)\}$，且

$$
f(x) = \sum_{n=-\infty}^{\infty} g(x-n) = \cdots + g(x+1) + g(x) + g(x-1) + \cdots
$$

$f(x)$ 如下圖



故 $f(x)$ 爲最小週期 1 的函數，又因

$$
g(x+1) = (x+1)^2\{H(x+1) - H(x-1)\} = \begin{cases} (x+1)^2 & ; -1 \le x < 1 \\ 0 & ; \text{otherwise} \end{cases}
$$

$$
g(x) = x^2\{H(x) - H(x-2)\} = \begin{cases} x^2 & ; 0 \le x < 2 \\ 0 & ; \text{otherwise} \end{cases}
$$

故在 $0 < x < 1$ 時

$$
f(x) = (x+1)^2 + x^2 = 2x^2 + 2x + 1
$$

因此 $f(x) = 2x^2 + 2x + 1$； $(0 \le x \le 1)$， $f(x) = f(x+1)$

(3) 週期函數與非週期函數的 convolution：設

$$
f(x) = g(x)[H(x) - H(x-T_0)] = \begin{cases} g(x) & ; 0 \le x < T_0 \\ 0 & ; \text{otherwise} \end{cases}
$$

且

$$
d(x) = \sum_{k=-\infty}^{\infty} \delta(x-kT) \; ; \; T > T_0 > 0
$$

令

$$
h(x) = f(x) * d(x) = \{g(x)[H(x) - H(x-T_0)]\} * \left(\sum_{k=-\infty}^{\infty} \delta(x-kT)\right)
$$

(a) 當 $0 < x < T_0$ 時

$$h(x) = \int_{x-T_0}^{x-0} d(\tau)f(x-\tau)\,d\tau = \int_{x-T_0}^{x} \delta(\tau)f(x-\tau)\,d\tau = f(x) = g(x)$$

(b) 當 $T_0 < x < T$

$$h(x) = \int_{x-T_0}^{x-0} d(\tau)f(x-\tau)\,d\tau = \int_{x-T_0}^{x} 0\cdot f(x-\tau)\,d\tau = 0$$

(c) 故 $h(x)$ 爲基本週期 T 的函數，且

$$h(x) = \begin{cases} g(x) & ;\ 0 < x < T_0 \\ 0 & ;\ T_0 < x < T \end{cases} \quad ;\ h(x) = h(x+T)$$

(d) 例子：

(i) 週期函數與非週期函數的 convolution：Let $f(x) = \dfrac{\pi - x}{2}$ and

$$g(x) = \{f(x)[H(x) - H(x-\pi)]\} * \left(\sum_{n=-\infty}^{\infty} \delta(x - 2n\pi)\right)$$

當 $0 < x < \pi$ 時

$$g(x) = \int_{x-\pi}^{x} \delta(\tau)f(x-\tau)\,d\tau = f(x) = \frac{\pi - x}{2}$$

當 $\pi < x < 2\pi$

$$g(x) = \int_{x-\pi}^{x} 0\cdot f(x-\tau)\,d\tau = 0$$

故 $g(x)$ 爲基本週期 2π 的函數。即

$$g(x) = \begin{cases} \dfrac{\pi - x}{2} & ;\ 0 < x < \pi \\ 0 & ;\ \pi < x < 2\pi \end{cases} \quad ;\ g(x) = g(x+2\pi)$$

(ii) Let $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$，$f(t) = H(t) - H(t-T_0)$，$0 < T_0 < T$

$$g(t) = x(t) * f(t) = f(t) * x(t) = \begin{cases} 1 & ;\ 0 < t < T_0 \\ 0 & ;\ T_0 < t < T \end{cases} \quad ;\ g(t) = g(t+T)$$

2. 性質：

(a) 若 $f(x)$ 爲週期 T 的函數，則

$$\int_a^{a+T} f(x)dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx$$

《 **題型 11** 》一般函數的 Fourier 級數

(1) 已知 $f(x)=e^{ax}$，$-\pi<x<\pi,\ a\neq 0$。試求 $f(x)$ 的傅立葉級數 (Fourier series)。(10%)

(2) 若 $g(x)=\cosh ax$，$-\pi<x<\pi,\ a\neq 0$。利用 (1) 之結果，試求 $g(x)$ 的傅立葉級數。(10%) **《喻超凡、喻超弘** 108 台大生機》

《解》

(1) 因 $f(x)$ 為不奇不偶的函數，故

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

其中

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ax}\,dx=\frac{e^{a\pi}-e^{-a\pi}}{2a\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx=\frac{a\cos n\pi\cdot(e^{a\pi}-e^{-a\pi})}{(a^2+n^2)\pi}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\sin nx\,dx=\frac{n\cos n\pi\cdot(e^{-a\pi}-e^{a\pi})}{(a^2+n^2)\pi}$$

(2)

$$\begin{aligned}g(x)&=\cosh ax=\frac{1}{2}(e^{ax}+e^{-ax})=\frac{1}{2}\{f(x)+f(-x)\}\\&=\frac{1}{2}\{\Big[a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\Big]+\Big[a_0+\sum_{n=1}^{\infty}(a_n\cos nx-b_n\sin nx)\Big]\}\\&=a_0+\sum_{n=1}^{\infty}a_n\cos nx\end{aligned}$$

其中

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ax}\,dx=\frac{e^{a\pi}-e^{-a\pi}}{2a\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{ax}\cos nx\,dx=\frac{a\cos n\pi\cdot(e^{a\pi}-e^{-a\pi})}{(a^2+n^2)\pi}$$

(a) 因 $f(x)$ 爲週期 2π 的偶函數，令

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

其中

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1\,dx = \frac{1}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos nx\,dx = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

故

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx$$

(b) ODE 的解爲

$$\begin{aligned} y(x) &= \frac{1}{5D^4+4}\{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx\} \\ &= \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\{\frac{1}{5D^4+4}\}\cos nx \\ &= \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\frac{\cos nx}{5n^4+4} \end{aligned}$$

《 **題型 32** 》一般函數的 Fourier 級數

Consider the square wave that is defined as

$$f(x) = \begin{cases} 0 & , (2n-1)\pi < x < 2n\pi \\ 4 & , 2n\pi < x < (2n+1)\pi \\ 2 & , x = n\pi \end{cases}$$

(1) Find the Fourier series. (10%)

(2) Using above series to show that $\frac{\pi}{4} = \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}$. (5%)

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《**解**》

(1) 當 $n=0$ 時，$f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ 4 & , 0 < x < \pi \end{cases}$，且 $f(x) = f(x+2\pi)$，令

$$g(x) = f(x) - 2 = \begin{cases} -2 & , -\pi < x < 0 \\ 2 & , 0 < x < \pi \end{cases}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} f(t)(\cos\omega t - i\,\sin\omega t)\,dt \\
&= 2\int_0^1 \frac{1}{2}\cos\omega t\,dt = \frac{\sin\omega}{\omega}
\end{aligned}
$$

《 題型 71 》Fourier 轉換

(10%)) Given that

$$h(x) = u(x+1) - u(x-1)$$

where $u(x-a) = \begin{cases} 1 & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}$ is the Heaviside step function. Find the Fourier transform of the fol1owing function $f(x) = h(x)\,\cosh x$.

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《解》因

$$f(x) = h(x)\,\cosh x = \cosh x\{u(x+1) - u(x-1)\} = \begin{cases} \cosh x & ; -1 \leq x < 1 \\ 0 & ; \text{otherwise} \end{cases}$$

故

$$
\begin{aligned}
\mathscr{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx \\
&= \int_{-1}^{1} \cosh x(\cos wx - i\sin wx)\,dx \\
&= 2\int_0^1 \cosh x\cdot\cos wx\,dx \\
&= \frac{2}{1+w^2}(\sinh x\cdot\cos wx + \cosh x\cdot w\sin wx)\Big|_0^1 \\
&= \frac{2}{1+w^2}(\sinh 1\cdot\cos w + w\,\cosh 1\cdot\sin w)
\end{aligned}
$$

《 題型 72 》Fourier 轉換

Find the Fourier transform of the following Gaussia function (a is a positive real constant) by direct integration along the temporal axis. Details required. (20%)

$$f(x) = \exp(-a^2t^2)$$

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《解》

$$
\begin{aligned}
F(\alpha) &= \int_{-\infty}^{\infty} \exp\{-x^2/(4p^2)\}\ \exp(i\alpha x)\,dx \\
&= \int_{-\infty}^{\infty} \exp\{-\frac{1}{4p^2}[x^2 - i4p^2\alpha x]\}\,dx \\
&= \int_{-\infty}^{\infty} \exp\{-\frac{1}{4p^2}[x-(i2p^2\alpha)]^2 - p^2\alpha^2\}\,dx \\
&= \exp\{-p^2\alpha^2\}\cdot\int_{-\infty}^{\infty} \exp\{-\frac{1}{4p^2}[x-(i2p^2\alpha)]^2\}\,dx \\
&= \exp\{-p^2\alpha^2\}\cdot\sqrt{\frac{\pi}{\frac{1}{4p^2}}} = 2p\sqrt{\pi}\ \exp\{-p^2\alpha^2\}
\end{aligned}
$$

故 $s(p)=2p$，$t(p)=p^2$，選 (C)(E)

《 **題型 90** 》Fourier 轉換

(1) (3%) Find the continuous-time non-periodic signal $x(t)$ with its Fourier transform

$$X(jw)=\frac{1}{jw+500}$$

(2) (7%) Find the continuous-time non-periodic signal $x(t)$ with its Fourier transform

$$X(jw)=\frac{5(jw)-100}{(jw)^2+100(jw)-120000}$$

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《解》

(1) $x(t)=\mathscr{F}^{-1}\{\frac{1}{jw+500}\}=e^{-500t}H(t)$

(2)

$$
\begin{aligned}
x(t) &= \mathscr{F}^{-1}\{\frac{5(jw)-100}{(jw)^2+100(jw)-120000}\} \\
&= \mathscr{F}^{-1}\{-\frac{2}{300-jw}+\frac{3}{400+jw}\} \\
&= -2e^{300t}H(-t)+3e^{-400t}H(t)
\end{aligned}
$$

$$b_n = \int_0^1 y(t)\sin(n\pi t)\,dt = \int_0^1 e^{-2t}\sin(n\pi t)\,dt = \frac{\pi n(e^2 - \cos\pi n)}{e^2(\pi^2 n^2 + 4)}$$

《**題型 97**》Fourier 轉換

For two continuous-time non-periodic signals, $x(t)$ and $y(t)$, please prove the following Fourier properties:

(a) $\mathscr{F}\{x(t) * y(t)\} = X(jw)\cdot Y(jw)$

(b) $\mathscr{F}\{-jt\cdot x(t)\} = \dfrac{d}{dw}X(jw)$

Remark: $\mathscr{F}\{\ \ \}$ is continuous-time Fourier transform operator, $X(jw)$ and $Y(jw)$ are Fourier transform of $x(t)$ and $y(t)$, respectively.

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《**解**》

(a)

$$\begin{aligned}\mathscr{F}\{x(t)*y(t)\} &= \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,d\tau\Big]e^{-j\omega t}\,dt\\ &= \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} x(t-\tau)e^{-j\omega t}\,dt\Big]y(\tau)\,d\tau\\ &\quad (\text{令 } z = t-\tau\text{ , } dz = dt\ )\\ &= \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} x(z)\,e^{-j\omega(z+\tau)}\,dz\Big]y(\tau)\,d\tau\\ &= \{\int_{-\infty}^{\infty} x(z)\cdot e^{-j\omega z}\,dz\}\{\int_{-\infty}^{\infty} y(\tau)e^{-j\omega\tau}\,d\tau\}\\ &= X(j\omega)\cdot Y(j\omega)\end{aligned}$$

(b) 因

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

故

$$\begin{aligned}\frac{d}{dw}X(jw) &= \frac{d}{dw}\int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt\\ &= \int_{-\infty}^{\infty}(-jt)\cdot x(t)e^{-jwt}\,dt\\ &= \mathscr{F}\{-jt\cdot x(t)\}\end{aligned}$$

即

$$\mathscr{F}\{-jt\cdot x(t)\} = \frac{d}{dw}X(jw)$$

且

$$\mathscr{F}\{u_t(x\,,\,0)\} = U_t(w\,,\,0) = 0 = B(w)\cdot(aw)$$

故 $B(w)=0$，則

$$U(w\,,\,t) = A(w)\cos(awt) = A(w)\cdot\frac{1}{2}(e^{iawt}+e^{-iawt})$$

因 $\mathscr{F}^{-1}\{A(w)\} = f(x) = e^{-x^2}$，故

$$\begin{aligned} u(x\,,\,t) &= \mathscr{F}^{-1}\{U(w\,,\,t)\} = \frac{1}{2}\{e^{-(x+at)^2}+e^{-(x-at)^2}\} \\ &= \frac{1}{2}\{e^{-(x^2+2atx+a^2t^2)}+e^{-(x^2-2atx+a^2t^2)}\} \\ &= e^{(-x^2-a^2t^2)}\ \cosh(2atx) \end{aligned}$$

因此 $s(t)=-a^2t^2$，$r(t)=\pm 2at$，選 (A)(C)(E)

《 **題型 120** 》應用題

(15%) Derive solution of the one-dimensional heat equation over the half-line by the Fourier Sine Transform Method. Describe the conditions of $f(x)$ and T so the solution holds.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad 0<x<\infty\,,\,t>0$$

$$u(0\,,\,t) = T \text{ for t} > 0$$

$$u(x\,,\,0) = f(x)\ ,\ \text{ for } 0<\text{x}<\infty$$

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《**解**》對 PDE 兩端 x 變數取 Fourier sine 轉換

$$\mathscr{F}_s\{\frac{\partial u}{\partial t}\} = \mathscr{F}_s\{k\frac{\partial^2 u}{\partial x^2}\}$$

可得

$$\frac{dU}{dt} = -w^2kU(w\,,\,t)+kwu(0\,,\,t)$$

其中 $\mathscr{F}_s\{u(x\,,\,t)\} = U(w\,,\,t)$, 整理可得

$$\frac{dU}{dt}+kw^2U(w\,,\,t) = kwT$$