107 年

線性代數題庫班

講義勘誤檔案

喻超凡博士編

喻超凡翻轉教室

工數神父 facebook

1. (5%) Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$. Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is

(A) a point (B) a line (C) a plane (D) $\mathbb{R}^3$

《107台大電機 C》

《解》 ☞ 令

$$A = [\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \boldsymbol{v}_3] = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

因 $\det(A) = 4 \neq 0$，故 $\boldsymbol{v}_1$、$\boldsymbol{v}_2$、$\boldsymbol{v}_3$ 爲線性獨立。故 **span** $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} = \mathbb{R}^3$。

2. (5%) Which of the following statements are correct?

(a) A basis for a subspace is a linearly independent subset of the subspace that is as large as possible.

(b) If V is a subspace of dimension k, then every generating set for V contains exactly k vectors.

(c) If $\beta = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}$ is any basis for $\mathbb{R}^n$, then there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ so that $\boldsymbol{v}$ can not be expressed as a linear combination of $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\cdots$, $\boldsymbol{b}_n$.

(d) Let T be a linear operator on $\mathbb{R}^n$, and μ and ν be two bases for $\mathbb{R}^n$. Denote $[T]_\beta$ as the representative matrix of T with respect to any basis β for $\mathbb{R}^n$. Then, $[T]_\mu$ and $[T]_\nu$ are similar.

(e) If β is a basis for $\mathbb{R}^n$ and $[T]$ is the identity operator on $\mathbb{R}^n$, then $[T]_\beta = I_n$.

《106台大電機 C》

《解》 ☞

(a) T；Steinitz's replacement theorem。基底爲向量空間中，最大的線性獨立子集合。

(b) F；不一定，例如 $\mathbb{R}^2 = \textbf{span}\{\vec{i}, \vec{i}+\vec{j}, \vec{j}\}$。

喻 喻 超 超 凡 弘

1. (20%) Let $E = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = [(1, 1, 1)^T, (2, 3, 2)^T, (1, 5, 4)^T]$ $F = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3] = [(1, 1, 0)^T, (1, 2, 0)^T, (1, 2, 1)^T]$. If $\boldsymbol{x} = 8\boldsymbol{u}_1 - 5\boldsymbol{u}_2 + 3\boldsymbol{u}_3$ and $\boldsymbol{y} = -8\boldsymbol{u}_1 + 2\boldsymbol{u}_2 + 3\boldsymbol{u}_3$, find the coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$ with respect to the ordered basis E. 《107中央電機固態》

《解》 ☞ 因

$$\boldsymbol{x} = 8\boldsymbol{u}_1 - 5\boldsymbol{u}_2 + 3\boldsymbol{u}_3$$

$$\boldsymbol{y} = -8\boldsymbol{u}_1 + 2\boldsymbol{u}_2 + 3\boldsymbol{u}_3$$

故

$$[\boldsymbol{x}]_F = \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix}, \quad [\boldsymbol{y}]_F = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$$

又有序基底 E 轉換到 $\mathbb{R}^3$ 的標準基 α 的轉換矩陣

$$[I]_E^\alpha = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

有序基底 F 轉換到 $\mathbb{R}^3$ 的標準基 α 的轉換矩陣

$$[I]_F^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

故有序基底 F 轉換到有序基底 E 的轉換矩陣

$$\begin{aligned} [I]_F^E &= [I]_\alpha^E \cdot [I]_F^\alpha = ([I]_E^\alpha)^{-1} \cdot [I]_F^\alpha \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -4 & -10 & -3 \\ 4 & 7 & 3 \\ -1 & -1 & 0 \end{bmatrix} \end{aligned}$$

故

$$[\boldsymbol{x}]_E = [I]_F^E \cdot [\boldsymbol{x}]_F = \frac{1}{3} \begin{bmatrix} -4 & -10 & -3 \\ 4 & 7 & 3 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$[\boldsymbol{y}]_E = [I]_F^E \cdot [\boldsymbol{y}]_F = \frac{1}{3}\begin{bmatrix} -4 & -10 & -3 \\ 4 & 7 & 3 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

2. Let $E := \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n\}$ be a basis of $\mathbb{R}^n$ and $F := \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\} \subset \mathbb{R}^n$ with the property $\forall i, j = 1, \cdots, n$, $\boldsymbol{u}_i^T\boldsymbol{v}_j = 1$ when $i = j$, and $\boldsymbol{u}_i^T\boldsymbol{v}_j = 0$ when $i \neq j$. Which of the following statements are true ?

(A) F is also a basis of $\mathbb{R}^n$, and the coordinate vector of any $\boldsymbol{x} \in \mathbb{R}^n$ with respect to base F is
$$\begin{bmatrix} \boldsymbol{x}^T\boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{x}^T\boldsymbol{v}_n \end{bmatrix}$$

(B) Any $\boldsymbol{x} \in \mathbb{R}^n$ can be represented as $\boldsymbol{x} = c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n$ with $c_i = \boldsymbol{x}^T\boldsymbol{u}_i$ for each i.

(C) Denote the transition matrix from base E to base F by S. Then $S(i, j) = \boldsymbol{u}_i^T\boldsymbol{u}_j$ for $i, j = 1, \cdots, n$.

(D) The transition matrix form base F to base E can be described by $([\boldsymbol{v}_1]_E \cdots [\boldsymbol{v}_n]_E)$, where $[\boldsymbol{v}_i]_E$ denotes the coordinate vector of $\boldsymbol{v}_i$ with respect to base E.

(E) Denote $U := [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_n]$ and $V := [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$. Then $U\boldsymbol{x} = \lambda\boldsymbol{x}$ for some $\boldsymbol{x} \neq 0$ if $(\overline{\boldsymbol{x}})^TV = (\overline{\lambda})^{-1}(\overline{\boldsymbol{x}})^T$, where the upper bar means to take the complex conjugate. 《106 中山電機乙》

《解》 ☞

(A) F;令 $U = [\boldsymbol{u}_1 \ \ \boldsymbol{u}_2 \ \ \cdots \ \ \boldsymbol{u}_n]$、$V = [\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \cdots \ \ \boldsymbol{v}_n]$,因

$$\begin{aligned} U^TV &= \begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_n^T \end{bmatrix}[\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \cdots \ \ \boldsymbol{v}_n] \\ &= \begin{bmatrix} \boldsymbol{u}_1^T\boldsymbol{v}_1 & \boldsymbol{u}_1^T\boldsymbol{v}_2 & \cdots & \boldsymbol{u}_1^T\boldsymbol{v}_n \\ \boldsymbol{u}_2^T\boldsymbol{v}_1 & \boldsymbol{u}_2^T\boldsymbol{v}_2 & \cdots & \boldsymbol{u}_2^T\boldsymbol{v}_n \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{u}_n^T\boldsymbol{v}_1 & \boldsymbol{u}_n^T\boldsymbol{v}_2 & \cdots & \boldsymbol{u}_n^T\boldsymbol{v}_n \end{bmatrix} \end{aligned}$$

1. (30分，計算題) Let $A = \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 2 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

Suppose P is a 4×4 matrix such that $P\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x}$ on Null A for every $\boldsymbol{x}$ in $\mathbb{R}^4$

(1) (5分) Is P unique ? Explain your answer. (No credits for answers without justifications.)

(2) (5分) Find the eigenvalues of P.

(3) (10分) Find the orthogonal projection of $\boldsymbol{b}$ on Null A.

(4) (5分) What is the distance between $\boldsymbol{b}$ and the orthogonal complement of Null A?

(5) (5分) Find a solution of $\boldsymbol{x}$ that minimizes $\|A^T\boldsymbol{x} - \boldsymbol{b}\|$.

《107台聯 D》

《解》 ☞

(1) 唯一，設 $\alpha = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$、$\beta = \{\boldsymbol{w}_1 . \boldsymbol{w}_2\}$ 均為 $\mathbf{N}(A)$ 的基底，再令 $U = WB$, 其中 $U = [\boldsymbol{u}_1 \ \ \boldsymbol{u}_1]$、$W = [\boldsymbol{w}_1 \ \ \boldsymbol{w}_2]$，$B$ 為從 α 變換到 β 的轉換陣，故

$$\begin{aligned} P &= U(U^TU)^{-1}U^T \\ &= (WB)((WB)^T(WB))^{-1}(WB)^T \\ &= WBB^{-1}(W^TW)^{-1}(B^T)^{-1}B^TW^T \\ &= W(W^TW)^{-1}W^T \end{aligned}$$

故 P 與 $\mathbf{N}(A)$ 的基底無關，即 P 唯一。

(2) 因 $\mathbf{N}(A)$ 的基底為 $\alpha = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$，其中

$$\boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

令 $U = [\boldsymbol{u}_1, \boldsymbol{u}_2]$，故

$$P = U(U^TU)^{-1}U^T$$

若 $\boldsymbol{v}_1 \in \mathbf{R}(U)^{\perp}$, 則 $\forall \boldsymbol{y} \in \mathbb{R}^2$, 使得

$$< \boldsymbol{v}_1 , U\boldsymbol{y} > = < U^T\boldsymbol{v}_1 , \boldsymbol{y} > = 0 ; \ \forall \boldsymbol{y} \in \mathbb{R}^2$$

故 $U^T\boldsymbol{v}_1 = \mathbf{0}$, 則

$$P\boldsymbol{v}_1 = U(U^TU)^{-1}U^T\boldsymbol{v}_1 = \mathbf{0}$$

因此 $\lambda_1 = 0$ 爲 P 的特徵值 , 又

$$\mathbf{rank}\,(P) = \mathbf{rank}\,\{U(U^TU)^{-1}U^T\} = \mathbf{rank}\,\{(U^TU)^{-1}U^T\} = \mathbf{rank}\,(U^T) = 2$$

故 P 有兩個 0 的特徵值, 即 $\lambda_2 = 0$, 再令 $\boldsymbol{v}_2 \in \mathbf{R}(U)$, 則 $\exists \boldsymbol{z} \in \mathbb{R}^2$ 使得 $U\boldsymbol{z} = \boldsymbol{v}_2$, 故

$$P\boldsymbol{v}_2 = U(U^TU)^{-1}U^T\boldsymbol{v}_2 = U(U^TU)^{-1}U^TU\boldsymbol{z} = U\boldsymbol{z} = \boldsymbol{v}_2$$

故 $\lambda_3 = 1$ 爲 P 的特徵值 , 又

$$\begin{aligned} tr(P) &= tr\{U(U^TU)^{-1}U^T\} = tr\{U^TU(U^TU)^{-1}\} = tr(I_{2\times 2}) = 2 \\ &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{aligned}$$

故 $\lambda_4 = 1$ 。

(3) 因 $\mathbf{N}(A)$ 的基底爲 $\alpha = \{\boldsymbol{u}_1 , \ \boldsymbol{u}_2\}$, 其中

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} , \ \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

設 α 對應的正交基底爲 $\beta = \{\boldsymbol{w}_1 , \boldsymbol{w}_2\}$, 其中 $\boldsymbol{w}_1 = \boldsymbol{u}_1$, 且

$$\boldsymbol{w}_2 = \boldsymbol{u}_2 - \frac{< \boldsymbol{u}_2 , \boldsymbol{w}_1 >}{< \boldsymbol{w}_1 , \boldsymbol{w}_1 >}\boldsymbol{w}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$$

故

$$\text{Proj}_{N(A)}\boldsymbol{b} = \frac{< \boldsymbol{b} , \boldsymbol{w}_1 >}{< \boldsymbol{w}_1 , \boldsymbol{w}_1 >}\boldsymbol{w}_1 + \frac{< \boldsymbol{b} , \boldsymbol{w}_2 >}{< \boldsymbol{w}_2 , \boldsymbol{w}_2 >}\boldsymbol{w}_2 = \frac{2}{2}\boldsymbol{w}_1 + 0\cdot\boldsymbol{w}_2 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(3)

$$\|\boldsymbol{w}_1\| = \left\| \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\| = \sqrt{2}$$

(4) 為求 $A^T\boldsymbol{x} = \boldsymbol{b}$ 最小二乘方的解，故

$$\boldsymbol{x} = (AA^T)^{-1}A\boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

2. (13%) Let V be a vector space.

(a) (3%) Let δ be a unit element of "+", the addition operation of V. (i) Write the equality condition about δ as shown in the corresponding axiom. (ii) Let γ be another unit element of "+". Show that $\delta = \gamma$.
(須註明使用到的所有向量空間定義中的公設, 否則不計分)

(b) (3%) Let X and Y be two subspaces of V. (i) Write the definition of $X + Y$. (ii) Write the definition of $X \oplus Y$. (iii) What is the mathematical relationship between $\mathbf{dim}\,(X+Y)$ and $\mathbf{dim}\,(X \oplus Y)$

(c) (2%) Let $< \bullet, \bullet >$ be an inner product defined on V. Show that the function $f(\boldsymbol{v}) := \sqrt{<\boldsymbol{v}, \boldsymbol{v}>}$ defined $\forall \boldsymbol{v} \in V$ satisfies the triangular inequality property.

(d) (5%) Consider the vector space $\mathbb{R}^{2\times 2}$ with the inner product $<A, B>= \text{trace}(\mathrm{A^T B})$ and denote $Y = \{A \in \mathbb{R}^{2\times 2} | A = A^T\}$. (i) Describe $Y^\perp$ as the span of an orthonormal basis. (ii) What is the matrix, denoted by $P_{Y^\perp}$, that represents the orthogonal projecting operation $\Pi_{Y^\perp} : \mathbb{R}^{2\times 2} \to Y^\perp$ along with the subspace Y with respect to the ordered basis $E = \{G, H\}$, where G and H are vectors of the standard bases for Y and $Y^\perp$, respectively?

《107中山電機乙》

《解》 ☞

(a) (i) 設 $\forall \boldsymbol{u}, \boldsymbol{v} \in V$，由公設的 $\boldsymbol{u} + \delta = \boldsymbol{u}$，可知 $\boldsymbol{u} + \delta = \boldsymbol{v} + \delta$，則 $\boldsymbol{u} = \boldsymbol{v}$。

(ii) 設 $\forall \boldsymbol{u} \in V$，由公設的 $\boldsymbol{u} + \delta = \delta + \boldsymbol{u} = \boldsymbol{u}$，可知

$$\gamma + \delta = \gamma \text{ 且 } \delta + \gamma = \gamma + \delta = \delta$$