101—102

工程數學歷屆試題

機械所 土木所

勘誤表

其中 $\mathscr{L}\{w(t)\} = W(s)$, 整理可得

$$-2sW(s) - s^2\frac{dW(s)}{ds} + (1-2n)sW(s) - \frac{d}{ds}W(s) = 0$$

□

$$\frac{dW(s)}{ds} + (1+2n)\frac{s}{s^2+1}W(s) = 0$$

故

$$\frac{dW(s)}{W(s)} = -(1+2n)\frac{s}{s^2+1}ds$$

兩端積分可得

$$\ln|W(s)| = -\frac{1}{2}(1+2n)\ln|s^2+1| + c_1$$

□

$$W(s) = \frac{c}{(s^2+1)^{\frac{1+2n}{2}}}$$

(c) 因

$$\begin{aligned} W(s) &= \frac{c}{(s^2+1)^{\frac{1+2n}{2}}} \\ &= c\{\frac{1}{s^{1+2n}}(1+\frac{1}{s^2})^{-(\frac{1}{2}+n)}\} \\ &= c\{\frac{1}{s^{1+2n}}\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}(\frac{1}{s^2})^m\} \\ &= c\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}\frac{1}{s^{1+2n+2m}} \end{aligned}$$

故

$$\begin{aligned} w(t) &= \mathscr{L}^{-1}\{W(s)\} \\ &= \mathscr{L}^{-1}\{c\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}\frac{1}{s^{1+2n+2m}}\} \\ &= c\sum_{m=0}^{\infty}\binom{-(\frac{1}{2}+n)}{m}\frac{t^{2m+2n}}{(2m+2n)!} \end{aligned}$$

因此

$$u(x,y) = (u_1 - u_0)y + u_0 + \sum_{n=1}^{\infty} \frac{2(-u_0 + u_1\cos n\pi)}{n\pi} e^{-n\pi x}\sin n\pi y$$

7. 已知□期□數 $f(t)$ 在某一□期內的定義爲

$$f(t) = 1 + t\,; \quad -1 < t < 1$$

(1) 試求□期□數 $f(t)$ 的□立葉級數 (Fourier series) 。(8%)

(2) 利用 $f(t)$ 的□立葉級數求出下列級數之和：

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(3) 假設 F 代表由 $f(t)$ 之□立葉級數定義於區間 $(-\infty, \infty)$ 的□數，試求 $F(1) + F(-5) - 3F(0) = ?$ (3%)　【□□□, □□□ 102台大生機】

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(1) 因

$$t = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

其中

$$b_n = 2\int_0^1 t\sin(n\pi t)\,dt = -\frac{2\cos n\pi}{n\pi} = \frac{(-1)^{n+1}2}{n\pi}$$

故

$$f(t) = 1 + t = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n\pi}\sin(n\pi t) \qquad (1)$$

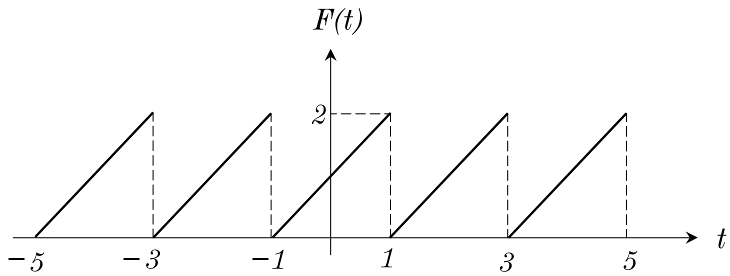
(2) 令 $t = \frac{1}{2}$ 代□ (1) 式中

$$1 + \frac{1}{2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n\pi}\sin\frac{n\pi}{2}$$

□

$$1 - \frac{1}{3} + \frac{1}{5} + \cdots = \frac{\pi}{4}$$

(3)



$F(1) + F(-5) - 3F(0) = 1 + 1 - 3 \times 1 = -1$

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1. (8%) Let

$$f(x) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^{96}}{96!} - \frac{x^{97}}{97!} - \frac{x^{98}}{98!} + \frac{x^{99}}{99!}$$

Find the closest value of maximum $f(x)$ for $x \in \mathbb{R}$ and $-3 < x < 3$.

(a) 0 (b) 1 (c) 2 (d) e (e) $\pi/2$ □□□□, □□□ 102交大機械甲□

□□□ ☞ 因

$$\begin{aligned} f(x) &= 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^{96}}{96!} - \frac{x^{97}}{97!} - \frac{x^{98}}{98!} + \frac{x^{99}}{99!} \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots - \frac{x^{98}}{98!}) - (x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots - \frac{x^{99}}{99!}) \\ &\approx \cos x - \sin x \\ &= \sqrt{2}(\cos\frac{\pi}{4}\cos x - \sin x \sin\frac{\pi}{4}) \\ &= \sqrt{2}\cos(x + \frac{\pi}{4}) \end{aligned}$$

當 $x = -\frac{\pi}{4}$ 時, 具有極大□爲 $f(\frac{\pi}{4}) \approx \sqrt{2}$, 故選 (e)

2. (8%) $(\sqrt{2i} - 1)^{2013} = ?$

(a) 0 (b) 1 (c) i (d) π (e) $2013 - 2013i$

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□□□ ☞ 因 $(1+i)^2 = 1 + 2i - 1 = 2i$, 故

$$(\sqrt{2i} - 1)^{2013} = (1 + i - 1)^{2013} = i^{2012} \times i = i$$

7. Please solve the following di erential equations :

(a) $\dfrac{dy}{dx} = \dfrac{1}{x + y^2}$, subject to $y(-2) = 0$ (7%)

(b) $y = 2x(y)^2$ (7%)

(c) Given that $y_1 = x^3$ is a solution of $x^2 y - 6y = 0$. Use reduction of order to find a second solution on the interval $0 < x < \infty$. (9%)

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(a) 原式可改□成

$$\frac{dx}{dy} = x + y^2 \qquad \frac{dx}{dy} - x = y^2$$

積分因子爲 $I = e^{-y}$, 故

$$I\,x = \quad y^2 e^{-y}\,dy = e^{-y}(-y^2 - 2y - 2) + c$$

因此

$$x = -y^2 - 2y - 2 + ce^{y}$$

再由 $y(-2) = 0$, 可得 $c = 0$, 故

$$x = -y^2 - 2y - 2$$

(b) 令 $p = y$, 代□ ODE 中可得

$$p = 2xp^2 \qquad \frac{dp}{p^2} = 2x\,dx$$

兩端積分可得

$$-\frac{1}{p} = x^2 + c_1 \qquad p = y = -\frac{1}{x^2 + c_1}$$

故

$$y = - \quad \frac{1}{x^2 + c_1}dx = -\frac{1}{\sqrt{c_1}}\tan^{-1}\frac{x}{\sqrt{c_1}} + c_2$$

令 $c_1 = \sqrt{c_1}$, 整理可得

$$y = -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1} + c_2$$

令

$$S=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}}\\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}}\\ 0 & -\frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}}\end{bmatrix}$$

故 S 爲正交矩，且

$$D=S^{-1}AS=\begin{bmatrix}-3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 7\end{bmatrix}$$

4. Solve the problem using Laplace transform. (15%)

$$y''+4y'+4y=1+\delta(t-1)\,,\ y(0)=0\,,\ y'(0)=2$$

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【】☞ 對 ODE 的取 L–T 可得

$$s^2Y(s)-sy(0)-y'(0)+4\{sY(s)-y(0)\}+4Y(s)=\frac{1}{s}+e^{-s}$$

其中 $\mathscr{L}\{y(t)\}=Y(s)$，整理可得

$$(s^2+4s+4)Y(s)=2+\frac{1}{s}+e^{-s}$$

$$Y(s)=\frac{2s+1}{s(s+2)^2}+\frac{1}{(s+2)^2}e^{-s}=\frac{1}{4s}+\frac{3}{2(s+2)^2}-\frac{1}{4(s+2)}+\frac{1}{(s+2)^2}e^{-s}$$

故

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\frac{1}{4}+\frac{3}{2}te^{-2t}-\frac{1}{4}e^{-2t}+(t-1)e^{-2(t-1)}H(t-1)$$

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$$I = \int_C 3x^2dx + 2yzdy + y^2dz = \int_{(0,1,2)}^{(1,-1,3)} d(x^3 + y^2z) = (x^3 + y^2z)\Big|_{(0,1,2)}^{(1,-1,3)} = 2$$

3. (17%) Find the Fourier transform of $f(x) = 1$ if $|x| = 2$ and $f(x) = 0$ otherwise.

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$$f(x) = \begin{cases} 1 & ; |x| \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

故

$$\begin{aligned}
\mathscr{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx \\
&= \int_{-2}^{2} (\cos wx - i\sin wx)\,dx \\
&= 2\int_0^2 \cos wx\,dx = 2\frac{\sin wx}{w}\Big|_0^2 \\
&= \frac{2\sin 2w}{w}
\end{aligned}$$

4. A damped mass-spring system is actuated by an external force. The model of system is described by differential equation :

$$y'' + 5y' + 6y = 12u(t-1) - 6u(t-3)\,,\ y(0) = 0\,,\ y'(0) = 0$$

(1) (7%) Determine the response of the system.

(2) (7%) State the steady-state of the response on $t > 3$

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□□□ ☞ 原式可改□成

$$\{xD^2 + (x^2 - 3)D - 2x\}y = 0$$

可因式分解成

$$(xD - 3)(D + x)y = 0$$

令 $z = (D + x)y$, 故可得

$$(xD - 3)z = 0 \qquad xz' - 3z = 0$$

或

$$\frac{dz}{z} - \frac{3}{x}dx = 0$$

兩端積分可得

$$\ln|z| - 3\ln|x| = c_1$$

□ $z = c_1x^3$, 故

$$(D + x)y = z = c_1x^3$$

爲一□線性 , 積分因子爲 $I = e^{\int x\,dx} = e^{\frac{x^2}{2}}$, 故

$$Iy = \int c_1x^3 e^{\frac{x^2}{2}}dx = c_1 e^{\frac{x^2}{2}}(x^2 - 2) + c_2$$

□

$$y(x) = c_1(x^2 - 2) + c_2e^{\frac{-x^2}{2}}$$

3. $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(a) Find A^{-1} using the method of Gauss-Jordan elimination. (8%)

(b) Find the eigenvalues of A^{-1}. (8%)

(c) Find the eigenvalues of A^5. (7%)

(d) If $D = P^{-1}AP$ is a diagonal matrix, find the matrix P. (7%)

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(a) $\left[\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_{13}} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \end{array}\right]$

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1. Use the idea of matrix to solve the following system of linear equations. (15%)

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 0 \\ -x_1 + x_3 + x_4 + 2x_5 = 0 \\ x_2 + 3x_3 + 4x_5 = 0 \\ -3x_1 + x_2 + 4x_5 = 0 \end{cases}$$

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$$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ -3 & 1 & 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_{12}^{(1)}R_{14}^{(-3)}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 7 & 3 & 3 & 7 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 2 & 2 & 2 & 3 \\ 0 & 7 & 3 & 3 & 7 \end{bmatrix}$$

$$\xrightarrow{R_{23}^{(-2)}R_{24}^{(-7)}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & -4 & 2 & -5 \\ 0 & 0 & -18 & 3 & -21 \end{bmatrix} \xrightarrow{R_{34}^{(-9/2)}} \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & -6 & 3/2 \end{bmatrix}$$

令 $x_5 = 8c$，則 $x_4 = 2c$、$x_3 = -9c$、$x_2 = -5c$、$x_1 = 9c$，c 爲任意數。

2. The circuit has the output voltage $q(t)/C$ as the following equation. The current and charge on the capacitor are zero at time zero ($q(0) = q\ (0) = 0$). Please determine the output voltage response to transient modeled by $\delta(t)$. (15%)

$$q\ \ + 20q\ \ + 25q = \delta(t)$$

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【□□】☞ 對 ODE 兩端取 L–T，可得

$$s^2Q(s) - sq(0) - q\ (0) + 20\{sQ(s) - q(0)\} + 25Q(s) = 1$$

其中 $\mathscr{L}\{q(t)\} = Q(s)$, 整理可得

$$(s^2 + 20s + 25)Q(s) = 1$$

□

$$Q(s) = \frac{1}{s^2 + 20s + 25} = \frac{1}{(s + 10)^2 - 75}$$

故

$$q(t) = \mathscr{L}^{-1}\{Q(s)\} = e^{-10t}\frac{1}{\sqrt{75}}\sinh(\sqrt{75}t)$$

> 3. The point $A(1, -2, 1)$, $B(0, 1, 6)$, and $C(-3, 4, -2)$ from the vertices of a triangle. Please calculate the cosine of the angle between AB and BC. (15%)
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【□】☞ $\overrightarrow{AB} = (-1, 3, 5)$ 、$\overrightarrow{BC} = (-3, 3, -8)$, 故

$$\cos\theta = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\cdot\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{3 + 9 - 40}{\sqrt{35}\sqrt{82}} = -\frac{28}{\sqrt{2870}}$$

> 4. Please find the mass and center of mass of the wire. The wire is bent into the shape of the quarter circle C given by
>
> $$x = 2\cos t,\ y = \sin t,\ z = 3,\ \text{for } 0 \le t \le \frac{\pi}{2}$$
>
> The density function is $\delta(x, y, z) = xy^2$ grams/cm^3. (15%)
>
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【□】☞ C 的位置向量爲

$$\vec{r} = x\vec{i} + y\vec{j} = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$$

故

$$ds = \left|\frac{d\vec{r}}{dt}\right|dt = 2\,dt$$

則 C 的質量爲

$$m = \int_C \delta(x,y,z)\,ds = \int_C xy^2\,ds = \int_0^{\frac{\pi}{2}} (2\cos t)(2\sin t)^2\ 2\,dt = \frac{16}{3}\sin^3 t\Big|_0^{\frac{\pi}{2}} = \frac{16}{3}$$

$$\begin{aligned} M_x &= \int_C x\delta(x,y,z)\,ds = \int_C x^2y^2\,ds \\ &= \int_0^{\frac{\pi}{2}} (2\cos t)^2(2\sin t)^2\ 2\,dt = 16B(\frac{3}{2},\frac{3}{2}) \\ &= 16\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} = 16\,\frac{\frac{1}{2}\sqrt{\pi}\ \frac{1}{2}\sqrt{\pi}}{2} = 2\pi \end{aligned}$$

$$M_y = \int_C y\delta(x,y,z)\,ds = \int_C xy^3\,ds = \int_0^{\frac{\pi}{2}} (2\cos t)(2\sin t)^3\ 2\,dt = \frac{32}{4}\sin^4 t\Big|_0^{\frac{\pi}{2}} = 8$$

設質心爲 $(\bar{x}, \bar{y}, \bar{z})$, 則

$$\bar{x} = \frac{M_x}{m} = \frac{3\pi}{8},\quad \bar{y} = \frac{M_y}{m} = \frac{3}{2},\quad \bar{z} = 3$$

5. This is non-autonomous, where f and g have explicit $t-$ dependencies, Please solve the system. (15%)

$$x'(t) = \frac{1}{t}x = f(t,x,y),\quad y' = -\frac{1}{t}y + x = g(t,x,y)$$

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☞ 由

$$x'(t) = \frac{1}{t}x \Rightarrow \frac{dx}{x} = \frac{1}{t}\,dt$$

兩端積分, 可得

$$\ln|x| = \ln|t| + c_1 \Rightarrow x = c_1 t$$

再由

$$y' = -\frac{1}{t}y + x = -\frac{1}{t}y + c_1 t$$

整理可得

$$y' + \frac{1}{t}y = c_1 t$$

積分因子 $I = \exp\{\int \frac{1}{t} dt\} = \exp\{\ln|t|\} = t$, 故

$$Iy = \int t(c_1 t)\,dt = c_1\frac{t^3}{3} + c_2$$

則

$$y(t) = c_1\frac{t^2}{3} + c_2\frac{1}{t}$$

6. Imaging a particle moving along a path having position vector

$$\vec{F}(t) = \sin t\,\vec{i} + 2e^t\vec{j} + t^2\vec{k}$$

Please write the velocity, acceleration, and speed of the particle. (15%)

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□□□ ☞ velocity 爲

$$\vec{V} = \frac{d\vec{F}}{dt} = \cos t\,\vec{i} + 2e^t\vec{j} + 2t\vec{k}$$

acceleration

$$\vec{a} = \frac{d^2\vec{F}}{dt^2} = -\sin t\,\vec{i} + 2e^t\vec{j} + 2\vec{k}$$

speed 爲

$$|\vec{V}| = \sqrt{\cos^2 t + 4e^{2t} + 4t^2}$$

7. Solve the following differential equation. $y'' + 4y' + 3y = e^t$, $y(0) = 0$, $y'(0) = 2$. (10%)

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(1) 齊次解：令 $y = e^{mt}$ 代□ ODE 中可得

$$m^2 + 4m + 3 = 0 \quad \Rightarrow \quad m = -1, -3$$

則

$$u_h(t) = e^{-\alpha t}(c_1 \cos wt + c_2 \sin wt)$$

(c) $c^2 - 4mk = 0$ 臨界□□ (critical damping)

特□方程式的根爲重根，令

$$\lambda_1 = \lambda_2 = -\alpha = -\frac{c}{2m}$$

則

$$u_h(t) = e^{-\alpha t}(c_1 + c_2 t)$$

(2) 特解

$$\begin{aligned} u_p(t) &= \frac{1}{mD^2 + cD + k}\sin wt = \frac{1}{m(-w^2) + cD + k}\sin wt \\ &= \frac{cD - (k - mw^2)}{\{cD + (k - mw^2)\}\{cD - (k - mw^2)\}}\sin wt \\ &= \frac{cD - (k - mw^2)}{c^2D^2 - (k - mw^2)^2}\sin wt \\ &= \frac{1}{-c^2w^2 - (k - mw^2)^2}\{cw\cos wt - (k - mw^2)\sin wt\} \end{aligned}$$

(3) 通解爲：$u(t) = u_h(t) + u_p(t)$

2. (20%) Please find the principle stresses (Eigenvalue) and their orientation (Eigenvectors) given $\sigma_x = 60$, $\sigma_y = 100$, $\tau_{xy} = 20$.

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【□】☞ 令

$$A = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} 60 & 20 \\ 20 & 100 \end{bmatrix}$$

由

$$\det(A - \lambda I) = \lambda^2 - 160\lambda + 5600 = 0$$

故 A 的特□□爲 $\lambda = 80 \pm 20\sqrt{2}$，將 $\lambda = 80 + 20\sqrt{2}$ 代□ $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} -20 - 20\sqrt{2} & 20 \\ 20 & 20 - 20\sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5. (20%) Consider the function of complex variable

$$f(z) = \frac{e^{az}}{e^{z}+1}, \quad 0 < a < 1$$

(a) Locate the singularities and evaluate the residues of $f(z)$.

(b) Evaluate the following integral using the Residue Theorem

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{e^{x}+1}\,dx, \quad 0 < a < 1$$

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(a) 因 $e^{z}+1=0$，可得

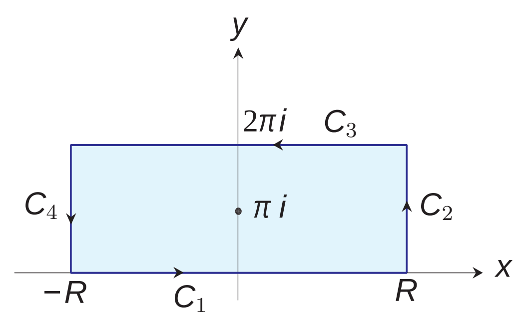
$$e^{z} = -1 = e^{(2n\pi+\pi)i} \quad (n = 0, \pm1, \pm2, \cdots)$$

□

$$z = (2n\pi+\pi)i \quad (n = 0, \pm1, \pm2, \cdots)$$

為 $f(z)$ 的一□ poles，且

$$\operatorname{Res} f((2n\pi+\pi)i) = \frac{e^{a(2n\pi+\pi)i}}{e^{(2n\pi+\pi)i}} = -e^{a(2n\pi+\pi)i} \quad (n = 0, \pm1, \pm2, \cdots)$$



(b) 因 $f(z) = \dfrac{e^{az}}{1+e^{z}}$，故 $f(z)$ 在 C ($C = C_1 + C_2 + C_3 + C_4$ 如圖) 內具有 $z = \pi i$ 的一□ pole，且

$$\operatorname{Res} f(\pi i) = \lim_{z\to\pi i} \frac{e^{az}}{e^{z}} = e^{(a-1)\pi i} = -e^{a\pi i}$$

□□□ ☞ 對 ODE 兩端取 L–T , 可得

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 1 - e^{-\pi s}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$, 整理可得

$$Y(s) = \frac{1}{s^2+4} - \frac{1}{s^2+4}e^{-\pi s}$$

故

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{2}\sin 2t - \frac{1}{2}\sin 2(t-\pi)\,H(t-\pi)$$

4. (20%) Let S be a piecewise smooth closed surface bounding a region V. Show that volume of $V = \frac{1}{3}\oint\!\!\oint_S (\vec{r}\cdot\vec{n})\,dS$. Where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

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□□□ ☞ 因

$$\oint\!\!\oint_S (\vec{r}\cdot\vec{n})\,dS = \iiint_V (\nabla\cdot\vec{r})\,dV = \iiint_V 3\,dV$$

故

$$V = \iiint_V dV = \frac{1}{3}\oint\!\!\oint_S (\vec{r}\cdot\vec{n})\,dS$$

5. (20%) For a circular membrane subject to a pressure $p(x, y) = x$, the mathematic model is

$$\nabla^2 u = x\,,\quad u = 0 \text{ for } x^2 + y^2 = a$$

Find the solution $u(x, y)$.　　□□□□, □□□ 102成大土木□

□□□ ☞ 因區□爲圓的區□ , 故 $u(r,\theta) = u(r,\theta+2\pi)$, 則特□□數爲

$$\left\{1,\ \cos n\theta,\ \sin n\theta\right\}_{n=1}^{\infty}$$

因此由特□□數展開法知 , 令

$$u(r,\theta) = a_0(r) + \sum_{n=1}^{\infty}\{a_n(r)\cos n\theta + b_n(r)\sin n\theta\}$$

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1. (10%) Find all solution or indicate the no solution exists.

$$\begin{cases} 4y + z = 0 \\ 12x - 5y - 3z = 34 \\ -6x + 4z = 8 \end{cases}$$

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□□□ ☞ 原式可改□成

$$\begin{bmatrix} 0 & 4 & 1 \\ 12 & -5 & -3 \\ -6 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 34 \\ 8 \end{bmatrix}$$

令

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 12 & -5 & -3 \\ -6 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 34 \\ 8 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 12 & -5 & -3 & 34 \\ -6 & 0 & 4 & 8 \end{array}\right] \xrightarrow{R_{32}^{(2)}} \left[\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 0 & -5 & 5 & 50 \\ -6 & 0 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_{2}^{(1/5)}} \left[\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 0 & -1 & 1 & 10 \\ -6 & 0 & 4 & 8 \end{array}\right] \xrightarrow{R_{21}^{(4)}} \left[\begin{array}{ccc|c} 0 & 0 & 5 & 40 \\ 0 & -1 & 1 & 10 \\ -6 & 0 & 4 & 8 \end{array}\right]$$

故可解得 $z = 8$、$y = -2$、$x = 4$

2. (15%) Find the eigenvalues and the corresponding eigenvectors. Use the given λ.

$$A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}, \quad \lambda = 4$$

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□□□ ☞ 因

$$T = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k} = -\frac{2xz}{(x^2+y^2)^2}\vec{i} - \frac{2yz}{(x^2+y^2)^2}\vec{j} + \frac{1}{x^2+y^2}\vec{k}$$

故

$$T(P) = -4\vec{j} + \vec{k}$$

故熱流的方向爲 $-\frac{T(P)}{|T(P)|} = -\frac{-4\vec{j}+\vec{k}}{\sqrt{17}}$

4. Solve the following ordinary di erential equations (ODEs) :

(a) (10%) $(1-2x-x^2)y \ \ + 2(1+x)y \ \ - 2y = 0$

(b) (13%) $2xy \ \ + (1+x)y \ \ + y = 0$

(c) (12%) $y \ \ + 3y \ \ + 2y = \begin{cases} 0 & , \text{if } t<1 \\ 1 & , \text{if } 1<t<2 \\ 0 & , \text{if } t>2 \end{cases}$ with I.C. $\begin{cases} y(1) = 1 \\ y\ (1) = -1 \end{cases}$

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□□□ ☞

(a) 已知一解爲 $y_1 = (1+x)$, 故 ODE □外一解爲

$$\begin{aligned} y_2 &= (1+x)\int \frac{e^{-\int \frac{2(1+x)}{(1-2x-x^2)}dx}}{(1+x)^2}dx \\ &= (1+x)\int \frac{e^{\ln|x^2+2x-1|}}{(1+x)^2}dx \\ &= (1+x)\int \frac{x^2+2x-1}{(1+x)^2}dx \\ &= (1+x)\int \{1-\frac{2}{(1+x)^2}\}dx \\ &= (1+x)\{x+\frac{2}{1+x}\} \\ &= x(x+1)+2 = x^2+x+2 \end{aligned}$$

故 ODE 的通解爲

$$y(x) = c_1(x+1) + c_2(x^2+x+2)$$

(b) 原式可改□成

$$(2xy\ ) \ - 2y \ \ + (1+x)y \ \ + y = 0$$

□

$$(2xy')' + (x-1)y' + y = 0 \qquad (2xy')' + \{(x-1)y\}' = 0$$

兩端積分可得

$$2xy' + (x-1)y = c_1$$

□

$$y' + \frac{x-1}{2x}y = \frac{c_1}{2x}$$

積分因子爲

$$I = \exp\{\int \frac{x-1}{2x}dx\} = \exp\{\frac{x}{2} - \frac{1}{2}\ln|x|\} = \frac{e^{\frac{x}{2}}}{\sqrt{x}}$$

則

$$Iy = \int \frac{c_1}{2x}\frac{e^{\frac{x}{2}}}{\sqrt{x}}dx + c_2$$

故 ODE 的通解爲

$$y(x) = \frac{c_1}{2}\sqrt{x}e^{\frac{-x}{2}}\int \frac{e^{\frac{x}{2}}}{x\sqrt{x}}dx + c_2\sqrt{x}e^{\frac{-x}{2}}$$

(c) 令 $x = t-1$，故 $t = 1$ 時 $x = 0$，且

$$f(t) = f(x) = \begin{cases} 0 & , \text{if } x < 0 \\ 1 & , \text{if } 0 < x < 1 \\ 0 & , \text{if } x > 1 \end{cases} = H(x) - H(x-1)$$

且 ODE 可改□成

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x),\ y(0) = 1,\ \frac{dy}{dx}(0) = -1$$

故 $\mathscr{L}\{f(x)\} = \frac{1}{s}e^{-0s} - \frac{1}{s}e^{-s}$，對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - \frac{dy}{dx}(0) + 3\{sY(s) - y(0)\} + 2Y(s) = \frac{1}{s}e^{-0s} - \frac{1}{s}e^{-s}$$

其中 $\mathscr{L}\{y(x)\} = Y(s)$, 整理可得

$$(s^2 + 3s + 2)Y(s) = s + 2 + \frac{1}{s}e^{-0s} - \frac{1}{s}e^{-s}$$

故

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{1}{s(s+2)(s+1)}e^{-0s} - \frac{1}{s(s+2)(s+1)}e^{-s} \\ &= \frac{1}{s+1} + \{\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\}e^{-0s} - \{\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\}e^{-s} \end{aligned}$$

□

$$y(x) = \mathscr{L}^{-1}\{Y(s)\}$$
$$= e^{-x} + \{\frac{1}{2} - e^{-x} + \frac{1}{2}e^{-2x}\}H(x) - \{\frac{1}{2} - e^{-(x-1)} + \frac{1}{2}e^{-2(x-1)}\}H(x-1)$$

則

$$y(t) = e^{-(t-1)} + \{\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\}H(t-1) - \{\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\}H(t-2)$$

5. Please answer the following questions :
   (a) (5%) What is "Fourier series" used for ?
   (b) (5%) What is "orthogonality" regarding the Fourier series ?
   (c) (10%) Please find the Fourier series according to the following periodic rectangular wave.

$$F(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases} \quad , \ k \text{ isaconstant}$$

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【□ □ ☞

(a) Fourier 級數可用來表示任一□期□數，無論此□期□數是否連續。

(b) Fourier 級數□是 sine 與 cosine 正交□數所構成的無窮級數。

(c) 因 $F(x)$ 爲□期 4 的□ □數，故

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

其中

$$a_0 = \frac{1}{2}\int_0^2 F(x)\,dx = \frac{1}{2}\int_0^1 k\,dx = \frac{k}{2}$$

$$a_n = \frac{2}{2}\int_0^2 F(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

可得對應的特□向量爲

$$X_2 = c_2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \quad (c_2 = 0)$$

將 $\lambda = -\frac{2}{3}$ 代□ $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ 1 & \frac{1}{2} & 0 \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特□向量爲

$$X_3 = c_3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \quad (c_3 = 0)$$

(3) 令

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 3 & -2 \\ -13 & -2 & -1 \end{bmatrix}$$

則

$$\lim_{n \to \infty} \sin^n(A\pi) = \lim_{n \to \infty} P \begin{bmatrix} \sin^n(0\pi) & 0 & 0 \\ 0 & \sin^n(\frac{1}{2}\pi) & 0 \\ 0 & 0 & \sin^n(-\frac{2}{3}\pi) \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 6 & 3 & -2 \\ -13 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 6 & 3 & -2 \\ -13 & -2 & -1 \end{bmatrix}^{-1}$$

$$\boxed{= \frac{1}{84} \begin{bmatrix} 64 & 24 & 16 \\ 96 & 36 & 24 \\ -64 & -24 & -16 \end{bmatrix}}$$

(3) 通解：

$$y(x) = y_h(x) + y_p(x) = c_1e^{-x} + c_2e^{x/2} - x - 2$$

再由

$$\begin{cases} y(0) = 1 = c_1 + c_2 - 2 \\ y'(0) = 0 = -c_1 + \frac{1}{2}c_2 - 1 \end{cases}$$

可解得 $c_1 = \frac{1}{3}$、$c_2 = \frac{8}{3}$，故

$$y(x) = \frac{1}{3}e^{-x} + \frac{8}{3}e^{x/2} - x - 2$$

3. (10%) Use the Laplace Transform to solve the given initial-value problem.

$$y'' + y = \sin t,\ y(0) = 1,\ y'(0) = -1$$

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□□□ ☞ 對 ODE 兩端取 L–T，可得

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2+1}$$

其中 $\mathscr{L}\{y(t)\} = Y(s)$, 整理可得

$$(s^2+1)Y(s) = \boxed{s - 1 + \frac{1}{s^2+1}}$$

□

$$Y(s) = \frac{s-1}{s^2+1} + \frac{1}{(s^2+1)^2}$$

因 $\mathscr{L}^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$, 兩端對 a □分，可得

$$\mathscr{L}^{-1}\{\frac{-2a}{(s^2+a^2)^2}\} = -\frac{1}{a^2}\sin at + \frac{t}{a}\cos at$$

故

$$\mathscr{L}^{-1}\{\frac{1}{(s^2+a^2)^2}\} = \frac{1}{2a^3}\sin at - \frac{t}{2a^2}\cos at$$

5. (10%) 水由消防水管□出會□受水平力 F_1，大小 200lb，見圖 1，救火員必須施加多少的力 F_3, 使得水管能□朝向水平 45° 的方向？

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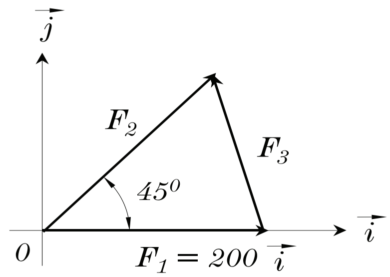

【□】 ☞ 由圖形可知，

$$\vec{F}_2 = 200\cos 45^\circ\ \vec{i} + 200\sin 45^\circ\ \vec{j} = 100\sqrt{2}\,\vec{i} + 100\sqrt{2}\,\vec{j}$$

則

$$\vec{F}_3 = \vec{F}_2 - \vec{F}_1 = 100\sqrt{2}\,\vec{i} + 100\sqrt{2}\,\vec{j} - 200\vec{i} = (100\sqrt{2} - 200)\vec{i} + 100\sqrt{2}\,\vec{j}$$

6. (10%) 給定矩□ A 為對稱，求出使得 A 對角化的正交矩□ P 及對角矩□ D，其中 $D = P^T AP$。

$$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

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【□】 ☞ 由 $\det(A - \lambda I) = 0$ 可得 A 的特□□為 $\lambda = 1$、8、−6, 將 $\lambda = 1$ 代□ $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & 0 \\ 7 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

可得對應的特□向量爲

$$X = c_1\begin{bmatrix}0\\1\\0\end{bmatrix} \quad (c_1 = 0) \quad 取 \quad X_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

將 $\lambda = 8$ 代□ $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix}-7 & 0 & 7\\0 & -7 & 0\\7 & 0 & -7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

可得對應的特□向量爲

$$X = c_2\begin{bmatrix}1\\0\\1\end{bmatrix} \quad (c_2 = 0) \quad 取 \quad X_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

將 $\lambda = -6$ 代□ $(A - \lambda I)X = 0$ 中可得

$$\begin{bmatrix}7 & 0 & 7\\0 & 7 & 0\\7 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

可得對應的特□向量爲

$$X = c_3\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad (c_3 = 0) \quad 取 \quad X_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

令

$$P = \left[\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 1 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

故

$$P^{-1}AP = D = \begin{bmatrix}1 & 0 & 0\\0 & 8 & 0\\0 & 0 & -6\end{bmatrix}$$

7. (10%) Find the curl and divergence of the given vector field. 求出給定向量場的□度和□度。

$$\vec{F}(x, y, z) = (x - y)^3\vec{i} + e^{-yz}\vec{j} + xye^{2y}\vec{k}$$

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2. (10%) Consider a homogenous linear 2nd order ordinary differential equation (ODE)

$$y'' + p(x)y' + q(x)y = 0$$

if y_1 is one solution of this ODE, you world find a second independent solution y_2 by reduction of order method, that is, $y_2 = uy_1$. Prove

$$u(x) = \int U(x)\,dx\,, \text{ and } U(x) = \frac{1}{y_1^2}e^{-\int p(x)\,dx}$$

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【□□】☞ 令 $y(x) = y_1(x)v(x)$, 故

$$y'(x) = y_1'(x)v(x) + y_1(x)v'(x)$$

$$y''(x) = y_1''(x)v(x) + 2y_1'(x)v'(x) + y_1(x)v''(x)$$

上式代入 ODE 式中

$$y_1''(x)v(x) + 2y_1'(x)v'(x) + y_1(x)v''(x) + p(x)\{y_1'(x)v(x) + y_1(x)v'(x)\}$$
$$+q(x)y_1(x)v(x) = 0$$

整理可得

$$y_1(x)v''(x) + \{p(x)y_1(x) + 2y_1'(x)\}v'(x) + \{y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)\}v(x) = 0 \quad (1)$$

因 $y_1(x)$ 為 ODE 的一解, 故

$$y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0$$

代□ (1) 式, 可得

$$y_1(x)v''(x) + \{p(x)y_1(x) + 2y_1'(x)\}v'(x) = 0$$

分離變數可得

$$\frac{dv'(x)}{v'(x)} = -\frac{p(x)y_1(x) + 2y_1'(x)}{y_1(x)}dx$$

將 $\lambda = 5$ 代於 $(A - \lambda I)X = 0$，中可得

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

對應的特徵向量爲

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

令

$$P = [X_1 \ \ X_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

故

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

則

$$A^{301} = PD^{301}P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & 5^{301} \end{bmatrix}P^{-1}$$

4. Solve the following differential equation by power series. (20%)

$$\frac{d^4y}{dx^4} + \sin x\,\frac{d^2y}{dx^2} = 0$$

Note : $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

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【解】☞ 因 $x = 0$ 爲 ODE 的常點，故令

$$y(x) = \sum_{n=0} \frac{y^{(n)}(0)}{n!}x^n$$

令 $y(0) = c_1$、$y'(0) = c_2$、$y''(0) = c_3$、$y'''(0) = c_4$，再由

$$y^{(4)}(x) = -(\sin x)y''(x) \qquad y^{(4)}(0) = 0$$

$$y^{(5)}(x) = -(\cos x)y''(x) - (\sin x)y'''(x) \qquad y^{(5)}(0) = -y''(0) = -c_3$$

$$y^{(6)}(x) = (\sin x)y''(x) - 2(\cos x)y'''(x) - (\sin x)y^{(4)}(x) \qquad y^{(6)}(0) = -2y'''(x) = -2c_4$$

$$\vdots$$