# 翻轉工程數學下冊

## 勘誤檔案

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## 習題解答

1. (a) $u_\infty(x) = \dfrac{100x}{L}$ (b) $u(\dfrac{L}{2}, (\dfrac{L}{5\alpha\pi})^2) = 100e^{-1} + 50$

2. $u(x, t) = \displaystyle\sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 kt} \sin n\pi x + \psi(x)$

 其中 $\psi(x) = -\dfrac{r}{2k}x^2 + (u_0 + \dfrac{r}{2k})x$ 、$A_n = 2\displaystyle\int_0^1 [f(x) - \psi(x)] \sin n\pi x \, dx$

3. $u(x, t) = \displaystyle\sum_{n=1}^{\infty} \frac{4L^2 - 2n^2\pi^2 T - 4L^2 \cos n\pi}{n^3\pi^3} e^{-\frac{n^2\pi^2 a^2}{L^2}t} \sin \frac{n\pi x}{L} + (-\frac{T}{L})x + T$

4. $u(x, t) = \displaystyle\sum_{n=1}^{\infty} \frac{6(-1)^n}{n\pi} e^{-(\frac{3n\pi}{5})^2 t} \sin \frac{n\pi x}{5} + \frac{3}{5}x$

5. $W(x, t) = \displaystyle\sum_{n=1}^{\infty} \{[-\frac{2}{n\pi} + \frac{2(-1)^n(-2 + 2n^2\pi^2 - n^2\pi^2 \sin 1)}{n\pi(n^2\pi^2 - 1)}] e^{-(n\pi)^2 t} \sin n\pi x\} + x + 1$

6. $u(x, y) = \displaystyle\sum_{n=1}^{\infty} \frac{2T(-1)^n}{n\pi} e^{-\frac{n\pi y}{a}} \sin \frac{n\pi x}{a} + \frac{T}{a}x$

7. $y(x, t) = \displaystyle\sum_{n=1}^{\infty} \frac{2AL^3(-1)^n}{n^3\pi^3} \cos \frac{n\pi t}{L} \sin \frac{n\pi x}{L} - \frac{Ax^3}{6} + \frac{AL^2 x}{6}$

8. $u(x, t) = e^{-a^2\pi^2 t} \sin \pi x + (\dfrac{1 - e^{-9a^2\pi^2 t}}{9a^2\pi^2}) \sin 3\pi x$

9 $u(x, t) = x$

10. $u(x, t) = -\dfrac{\ell}{2}e^{2t} + \displaystyle\sum_{n=1}^{\infty} \left[A_n e^{-(\frac{n\pi}{\ell})^2 t} - \frac{4\ell^3(\cos n\pi - 1)}{(2\ell^2 + n^2\pi^2)n^2\pi^2} e^{2t}\right] \cos \frac{n\pi x}{\ell} + xe^{2t}$

 其中 $A_n = \dfrac{4\ell^3(\cos n\pi - 1)}{(2\ell^2 + n^2\pi^2)n^2\pi^2} - \dfrac{2\ell}{n^2\pi^2}(\cos n\pi - 1)$

11. $u(x, t) = c_1 \operatorname{erf}(\dfrac{x}{2\sqrt{kt}}) + c_2$

12. $y(x, t) = y_0 \operatorname{erf}(\dfrac{x}{2\sqrt{at}})$

其中 $\mathscr{L}\{u(x,t)\} = U(x,s)$，整理可得

$$\frac{d^2U}{dx^2} - (\frac{s}{c})^2 \, U(x\,,\, s) = 0$$

上式的解爲

$$U(x\,,\, s) = k_1 e^{-\frac{s}{c}x} + k_2 e^{\frac{s}{c}x}$$

由 $u(\infty\,,\, t) =$ 有限值，故 $U(\infty\,,\, s) =$ 有限值，可得 $k_2 = 0$，又由 $u(0\,,\, t) = f(t)$，可得

$$U(0\,,\, s) = k_1 = \mathscr{L}\{f(t)\} = F(s)$$

因此

$$U(x\,,\, s) = F(s)\, e^{-\frac{x}{c}s}$$

則

$$u(x,t) = \mathscr{L}^{-1}\{U(x\,,\, s)\} = \mathscr{L}^{-1}\{F(s)\, e^{-\frac{x}{c}s}\} = f(t-\frac{x}{c})\, H(t-\frac{x}{c})$$

其中 $H(t)$ 爲 unit step function。

**9.** Find the solution $u(x,t)$ of the following partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \delta(x-at)\ ,\ c > a > 0\ ,\ 0 \le x < \infty\ ;\ 0 \le t < \infty$$

st. $u(x,0) = 0,\ \dfrac{\partial u(x,0)}{\partial t} = 0,\ u(0,t) = 0,\ u(x,t) < \infty$ as $x \to \infty$

Note : $\delta$ denotes the Dirac delta function. 《交大機械》

**《解》** ☞ 因

$$\delta(x-at) = \delta(at-x) = \frac{1}{a}\delta(t-\frac{x}{a})$$

故

$$\mathscr{L}\{\delta(x-at)\} = \frac{1}{a}\mathscr{L}\{\delta(t-\frac{x}{a})\} = \frac{1}{a}e^{-\frac{x}{a}s}$$

再對 PDE 中的 $t$ 變數取 Laplace 變換，可得

$$\frac{d^2}{dx^2}U(x,s) - \frac{1}{c^2}\{s^2U(x,s) - su(x,0) - u_t(x,0)\} = \frac{1}{a}e^{-\frac{x}{a}s}$$

其中 $\mathscr{L}\{u(x,t)\} = U(x,t)$，整理可得

$$\frac{d^2}{dx^2}U(x,s) - \frac{s^2}{c^2}U(x,s) = \frac{1}{a}e^{-\frac{x}{a}s} \tag{1}$$

故

$$U_h = k_1 e^{\frac{s}{c}x} + k_2 e^{-\frac{s}{c}x}$$
$$U_p = \frac{1}{D^2 - \frac{s^2}{c^2}}\left(\frac{1}{a}e^{-\frac{s}{a}x}\right) = \left(\frac{1}{a}\right)\frac{1}{(\frac{s}{a})^2 - (\frac{s}{c})^2}e^{-\frac{s}{a}x}$$

則 (1) 式的通解爲

$$U(x,s) = U_h + U_p = k_1 e^{\frac{s}{c}x} + k_2 e^{-\frac{s}{c}x} + \frac{ac^2}{s^2(c^2 - a^2)}e^{-\frac{s}{a}x}$$

由 $U(\infty,s) =$ 有限值，可得 $k_1 = 0$，由

$$U(0,s) = \mathscr{L}\{u(0,t)\} = 0 = k_2 + \frac{ac^2}{s^2(c^2 - a^2)}$$

可得 $k_2 = -\frac{ac^2}{s^2(c^2 - a^2)}$，故

$$U(x,s) = \frac{ac^2}{c^2 - a^2}\left(-\frac{1}{s^2}e^{-\frac{x}{c}s} + \frac{1}{s^2}e^{-\frac{x}{a}s}\right)$$

因此 PDE 的解爲

$$u(x,t) = \mathscr{L}^{-1}\{U(x,s)\} = \frac{ac^2}{c^2 - a^2}\Big[-(t - \frac{x}{c})\,H(t - \frac{x}{c}) + (t - \frac{x}{a})\,H(t - \frac{x}{a})\Big]$$

**10.** Solve the partial differential equation

$$\frac{\partial^3 u}{\partial t^3} = \frac{\partial u}{\partial x}$$

where $u$ is a function of $t$ and $x$ satisfying $u(t, x = -\infty) = 0$, $u(t = 0, x) = 0$.

$\frac{\partial u}{\partial t}\Big|_{t=0} = 0$ and $\frac{\partial^2 u}{\partial t^2}\Big|_{t=0} = e^{8x}$. 《清大電機》

**《解》** ☞ 對 PDE 中的 $t$ 變數取 Laplace 變換，可得

$$s^3 U(s,x) - s^2 u(0,x) - s u_t(0,x) - u_{tt}(0,x) = \frac{d}{dx}U(s,x)$$

**《解》** ☞ 兩端對 $t$ 取 Laplace 轉換，可得

$$sU(y,s)-u(y,0)=\nu\frac{d^2}{dy^2}U(y,s)$$

其中 $\mathscr{L}\{u(y,t)\}=U(y,s)$，整理可得

$$\frac{d^2}{dy^2}U(y,s)-\frac{s}{\nu}U(y,s)=0$$

故

$$U(y,s)=c_1\exp\{-\sqrt{\frac{s}{\nu}}y\}+c_2\exp\{\sqrt{\frac{s}{\nu}}y\}$$

又

$$\mathscr{L}\{u(\infty,t)\}=U(\infty,s)=0\quad\Rightarrow\quad c_2=0$$

$$\mathscr{L}\{u(0,t)\}=U(0,s)=-\mathscr{L}\{U_B(t)\}=-\hat{U}_B(s)=c_1$$

故

$$U(y,s)=-\hat{U}_B(s)\exp\{-\frac{y}{\sqrt{\nu}}\sqrt{s}\}$$

則

$$u(y,t)=\mathscr{L}^{-1}\{U(y,s)\}=-U_B(t)*(\frac{y}{2\sqrt{\nu\pi}}t^{-\frac{3}{2}}e^{-\frac{y^2}{4\nu t}})$$

**12.** $\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial t}=0\quad u(x\,,\,0)=0\,,\ u(0\,,\,t)=4t.$ 　　《交大土木、大同機械》

**《解》** ☞ 對 PDE 中的 $t$ 變數取 Laplace 變換，可得

$$\frac{dU}{dx}+x\{sU(x,s)-u(x\,,\,0)\}=0$$

其中 $\mathscr{L}\{u(x,t)\}=U(x,s)$，將 $u(x,0)$ 代入並整理得

$$\frac{dU}{dx}+xsU=0 \tag{1}$$

(1) 式為一階線性 ODE，其 I.C. 為

$$U(0\,,\,s)=\mathscr{L}\{u(0\,,\,t)\}=\mathscr{L}\{4t\}=\frac{4}{s^2}$$

求解 (1) 式可得

$$U(x,s)=ce^{-\frac{1}{2}sx^2}$$

## 習題解答

1. $u(x,t) = 2(e^{-5|x-3t|} + e^{-5|x+3t|})$

2. $u(x\,,\,t) = H(t - \frac{x}{a})f(t - \frac{x}{a})$

3. $u(x,t) = (B - A)\,\mathrm{erfc}(\frac{x}{2\sqrt{t}}) - B\,\mathrm{erfc}(\frac{x}{2\sqrt{t - t_0}})H(t - t_0) + A\,H(t)$

4. $$u(x,t) = \sum_{n=1}^{\infty}\Big\{\Big[(-1)^n\frac{2}{n\pi} - \frac{1}{8}\delta(n-3) + \frac{2}{\pi}\int_0^{\pi} f(x)\sin(n\pi x)dx\Big]e^{-n^2 t}$$
$$-\frac{2(-1)^n}{n\pi} + \frac{1}{8}e^{-t}\delta(n-3)\Big\}\sin(n\pi x) \quad (0 < x < \pi)$$

5. $u(x,t) = \dfrac{1}{6\sqrt{\pi t}}e^{-\frac{x^2}{36t}}$

6. $u(x,y) = \dfrac{2}{\pi}\sum_{n=1}^{\infty}\Big\{\dfrac{1}{n}[5 - 2(-1)^n]e^{-ny} - \dfrac{3}{n}(-1)^n\Big\}\sin nx$

**《解》** ☞ 特徵方程式爲

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{z}$$

由 $dx=dy$, 可得 $x-y=c_1$, 由 $dy=\dfrac{dz}{z}$ 可得 $\ln|z|-y=c_2$, 故 PDE 的通解爲

$$\ln|z|-y=f(x-y)$$

由 $z(x,0)=\dfrac{x}{2}+e^x$, 可得

$$\ln|\frac{x}{2}+e^x|=f(x)$$

故 PDE 的特解爲

$$\ln z-y=f(x-y)=\ln|\frac{x-y}{2}+e^{x-y}|$$

或

$$z=(\frac{x-y}{2}+e^{x-y})e^y=\frac{x-y}{2}e^y+e^x$$

**5.** Please find the general solution of the PDE $y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial y}=z-1$.
《台大工科、雲科大光電》

**《解》** ☞ PDE 的特徵方程式爲

$$\frac{dx}{y}=\frac{dy}{x}=\frac{dz}{z-1}$$

(1) 由 $\dfrac{dx}{y}=\dfrac{dy}{x}$ 可得

$$x\,dx=y\,dy\ \Rightarrow\ x^2-y^2=c_1$$

(2) 由 $\dfrac{dx+dy}{y+x}=\dfrac{dz}{z-1}$ 可得

$$(x+y)=c_2(z-1)\ \Rightarrow\ \frac{(x+y)}{z-1}=c_2$$

(3) 故 PDE 的通解爲

$$\frac{x+y}{z-1}=f(x^2-y^2)$$

**6.** Solve the following differential equations $\dfrac{1}{\ln x}\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=u$. Subject to the auxiliary condition $u(1\,,\,y)=y^2$.
《中興化工》

**7.** Consider the one dimensional wave propagation

$$u_{xx} = u_{tt}\ ;\quad -\infty < x < \infty\ ,\ ,t > 0$$

with initial conditions $u(x\,,\,0) = 0$ and $u_t(x\,,\,0) = g(x)$, where $g(x)$ is a given function.

(a) Show that $u(x\,,\,t) = G(x+t) - G(x-t)$ satisfies the above wave equation and initial conditions for a suitable function $G(x)$. How are $G(x)$ and $g(x)$ related?

(b) Find $u(x\,,\,t)$ if $u_t(x\,,\,0) = g(x) = \dfrac{x}{1+x^2}$ . 《台大機械》

**《解》** ☞

(a) 因 $u(x\,,\,t) = G(x+t) - G(x-t)$ , 故

$$\frac{\partial^2 u}{\partial x^2} = G''(x+t) - G''(x-t) \tag{1}$$

且

$$\frac{\partial^2 u}{\partial t^2} = G''(x+t) - (-1)^2 G''(x-t) = G''(x+t) - G''(x-t) \tag{2}$$

由 (1) 、(2) 兩式, 可知 $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, 即

$$u(x\,,\,t) = G(x+t) - G(x-t)$$

爲 PDE 的解, 因

$$\begin{cases} u(x\,,\,0) = 0 = G(x) - G(x) \\ u_t(x\,,\,0) = g(x) = G'(x) + G'(x) = 2G'(x) \end{cases}$$

故 $G'(x) = \dfrac{1}{2}g(x)$, 即 $G(x) = \dfrac{1}{2}\displaystyle\int_a^x g(\xi)\,d\xi \boxed{+\,G(a)}\ ,\ a \in \mathbb{R}$ 。

(b) 因 $g(x) = \dfrac{x}{1+x^2}$, 故

$$\begin{aligned} u(x\,,\,t) &= G(x+t) - G(x-t) \\ &= \frac{1}{2}\int_a^{x+t} g(\xi)\,d\xi \boxed{+\,G(a)} - \frac{1}{2}\int_a^{x-t} g(\xi)\,d\xi \boxed{-\,G(a)} \\ &= \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi = \frac{1}{2}\int_{x-t}^{x+t} \frac{\xi}{1+\xi^2}\,d\xi \\ &= \frac{1}{2}\times\frac{1}{2}\ln|1+\xi^2|\Big|_{x-t}^{x+t} = \frac{1}{4}\ln\{\frac{1+(x+t)^2}{1+(x-t)^2}\} \end{aligned}$$

因 **rank** $(A-\lambda I)=1$，故 $\lambda=0$ 對應的特徵向量為

$$V=\begin{bmatrix}v_1\\v_2\\v_3\\v_4\\v_5\end{bmatrix}=c_1\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+c_2\begin{bmatrix}-3\\0\\1\\0\\0\end{bmatrix}+c_3\begin{bmatrix}-4\\0\\0\\1\\0\end{bmatrix}+c_4\begin{bmatrix}-5\\0\\0\\0\\1\end{bmatrix}$$

其中 $c_1$、$c_2$、$c_3$、$c_4$ 不全為 0。

(b) 由

$$AX=(XX^T)X=X(X^TX)=X\lambda_5=\lambda_5X$$

可得 $\lambda_5=55$ 其所對應的特徵向量為

$$V=c_5\,X=c_5\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\quad(c_5\neq 0)$$

**22.** 若 $A$ 為 $3\times 3$ 的方陣，且其特徵值為 $\alpha_i$ ( $i=1,2,3$ )，令

$$B_{6\times 6}=\begin{bmatrix}4A & 3A\\5A & 6A\end{bmatrix}$$

求 $B$ 的特徵值。

**《解》** ☞ 令 $A$ 的特徵值 $\alpha_i$ 對應的特徵向量為 $\boldsymbol{u}_i$，則 $A\boldsymbol{u}_i=\alpha_i\boldsymbol{u}_i$，再令 $B$ 的特徵值為 $\lambda$ 且對應的特徵向量為

$$\boldsymbol{v}=\begin{bmatrix}a\boldsymbol{u}_i\\b\boldsymbol{u}_i\end{bmatrix}$$

故 $B\boldsymbol{v}=\lambda\boldsymbol{v}$，即

$$\begin{bmatrix}4A & 3A\\5A & 6A\end{bmatrix}\begin{bmatrix}a\boldsymbol{u}_i\\b\boldsymbol{u}_i\end{bmatrix}=\lambda\begin{bmatrix}a\boldsymbol{u}_i\\b\boldsymbol{u}_i\end{bmatrix}$$

則

$$\begin{bmatrix}4aA\boldsymbol{u}_i+3bA\boldsymbol{u}_i\\5aA\boldsymbol{u}_i+6bA\boldsymbol{u}_i\end{bmatrix}=\begin{bmatrix}4a\alpha_i\boldsymbol{u}_i+3b\alpha_i\boldsymbol{u}_i\\5a\alpha_i\boldsymbol{u}_i+6b\alpha_i\boldsymbol{u}_i\end{bmatrix}=\lambda\begin{bmatrix}a\boldsymbol{u}_i\\b\boldsymbol{u}_i\end{bmatrix}$$

# 精選範例

**1.** Please show that an $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors. 《成大電機》

**《解》** ☞ (1) 充分條件：設 $A$ 的特徵值 $\lambda_i$ 對應的線性獨立特徵向量爲 $X_i$，故

$$AX_i = \lambda_i X_i \quad (i = 1\,,\,2\,,\,\cdots n)$$

令 $S = [X_1 \quad X_2 \quad \cdots \quad X_n]$，則

$$\begin{aligned} AS &= A[X_1 \quad X_2 \quad \cdots \quad X_n] = [AX_1 \quad AX_2 \quad \cdots \quad AX_n] \\ &= [\lambda_1 X_1 \quad \lambda_2 X_2 \quad \cdots \quad \lambda_n X_n] \\ &= [X_1 \quad X_2 \quad \cdots \quad X_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = SD \end{aligned}$$

因 $X_1$、$X_2$、$\cdots$、$X_n$ 爲線性獨立, 則 $S^{-1}$ 存在，故 $S^{-1}AS = D$。

(2) 必要條件：因 $A$ 爲可對角化的矩陣，故存在一個非奇異矩陣 $S$，使得 $S^{-1}AS = D$，其中 $S = [X_1 \quad X_2 \quad \cdots \quad X_n]$，且

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

因此

$$\begin{aligned} AS &= A[X_1 \quad X_2 \quad \cdots \quad X_n] = [AX_1 \quad AX_2 \quad \cdots \quad AX_n] \\ &= SD = [\lambda_1 X_1 \quad \lambda_2 X_2 \quad \cdots \quad \lambda_n X_n] \end{aligned}$$

即

$$AX_i = \lambda_i X_i \quad (i = 1\ ,\ 2\ ,\ \cdots\ ,\ n)$$

故 $X_i$ 爲 $A$ 的特徵向量，又 $S$ 爲非奇異矩陣，則 $\{X_1\ ,\ X_2\ ,\ \cdots\ ,\ X_n\}$ 爲 $A$ 的線性獨立特徵向量。

取對應的線性獨立的特徵向量爲

$$Y_2=\begin{bmatrix}1\\1-\sqrt{2}i\\2+\sqrt{2}i\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}+i\begin{bmatrix}0\\-\sqrt{2}\\\sqrt{2}\end{bmatrix}=U+iV$$

且 $\lambda=-\sqrt{2}\,i$ 對應的線性獨立的特徵向量爲

$$Y_3=\begin{bmatrix}1\\1\\2\end{bmatrix}-i\begin{bmatrix}0\\-\sqrt{2}\\\sqrt{2}\end{bmatrix}=U-iV$$

(4) ODE 的解爲

$$X=c_1Y_1e^{2t}+\left[c_2(U\cos\sqrt{2}\,t-V\sin\sqrt{2}\,t)+c_3(U\sin\sqrt{2}\,t+V\cos\sqrt{2}\,t)\right]$$

由

$$X(0)=\begin{bmatrix}1\\0\\1\end{bmatrix}=c_1\begin{bmatrix}1\\-1\\-2\end{bmatrix}+c_2\begin{bmatrix}1\\1\\2\end{bmatrix}+c_3\begin{bmatrix}0\\-\sqrt{2}\\\sqrt{2}\end{bmatrix}$$

可得 $c_1=\dfrac{1}{3}$、$c_2=\dfrac{2}{3}$、$c_3=\dfrac{\sqrt{2}}{6}$

**16.** 求解 $X'=\begin{bmatrix}1&1&1\\2&1&-1\\-3&2&4\end{bmatrix}X$ 《中興化工、成大航太》

**《解》** ☞ 令

$$A=\begin{bmatrix}1&1&1\\2&1&-1\\-3&2&4\end{bmatrix}$$

由

$$\det(A-\lambda I)=-(\lambda^3-6\lambda^2+12\lambda-8)=0$$

可得 $A$ 的特徵值爲 $\lambda=2$、$2$、$2$。將 $\lambda=2$ 代回 $(A-\lambda I)V=0$ 中可得

$$\begin{bmatrix}-1&1&1\\2&-1&-1\\-3&2&2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

(3) ODE 的通解為

$$X = X_h + X_p = \alpha \begin{bmatrix} 3e^{-2t} \\ e^{-2t} \end{bmatrix} + \beta \begin{bmatrix} e^{-12t} \\ -3e^{-12t} \end{bmatrix} + \begin{bmatrix} \frac{276}{29}\sin t - \frac{168}{29}\cos t \\ \frac{332}{29}\sin t - \frac{76}{29}\cos t \end{bmatrix}$$

**5.** Find a general solution of the following systems of ODEs by the method of undetermined coefficients :

$$\begin{cases} y_1' = 2y_1 + 2y_2 + 5e^{4t} \\ y_2' = 5y_1 - y_2 - 2e^{4t} \end{cases}$$

《交大土木》

**《提示》** ➥ 利用待定係數法解聯立非齊次 ODE，但特解必須修正。

**《解》** ☞ ODE 改寫成

$$Y' = AY + B\,e^{-4t} \tag{1}$$

其中

$$A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix},\ B = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

由 $\det(A - \lambda I) = 0$, 可得 $A$ 的特徵值為 $\lambda = -3$ 、$4$，將 $\lambda = -3$ 代回 $(A - \lambda I)V = 0$ 中可得

$$\begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

故可得對應的特徵向量為

$$V_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

將 $\lambda = 4$ 代回 $(A - \lambda I)V = 0$ 中可得

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

故可得對應的特徵向量為

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

# 精選範例

**1.** Let $\vec{r}_1 = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$, $\vec{r}_2 = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$, $\vec{r}_3 = x_3\vec{i} + y_3\vec{j} + z_3\vec{k}$, be the position vectors of points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$. Find an equation for the plane passing through $P_1$, $P_2$, and $P_3$. 《交大機械甲》

《解》 ☞ 通過 $P_1$、$P_2$、$P_3$ 的平面的方程式爲

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

**2.** 求過點 $(1, 2, 3)$ 且平行於二向量 $\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$、$\vec{v} = 6\vec{i} + 3\vec{j} - 2\vec{k}$ 之平面方程式。 《中山材料》

《解》 ☞ 設平面的法向量爲 $\vec{n}$, 則

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 6 & 3 & -2 \end{vmatrix} = -\vec{i} - 4\vec{j} - 9\vec{k}$$

故平面方程式爲

$$-1 \cdot (x-1) - 4 \cdot (y-2) - 9 \cdot (z-3) = 0$$

即 $x + 4y + 9z = 36$。

令 $f(z)=\dfrac{2}{z^2(iz-2)}$, 故 $f(z)$ 在 $C$ 內具有 $z=0$ 的 2 階 pole, 且

$$\text{Res } f(0)=\lim_{z\to 0}\frac{d}{dz}\left(\frac{2}{iz-2}\right)=-\frac{i}{2}$$

因此

$$\oint_C \frac{\overline{z}}{\overline{z}-(i/2)}\,d\overline{z}=\oint_C \frac{2}{z^2(iz-2)}dz=2\pi i\,\text{Res } f(0)=2\pi i\left(-\frac{i}{2}\right)=\pi$$

**26.** Evaluate the complex integral $\oint_C z^2 \sin\overline{z}\,dz$ over the closed contour $C$ defined by $|z|=1$. 《台大機械》

《解》☞ 因 $C$ : $|z|=1$ , 由 $z\overline{z}=|z|^2=1$ , 故 $\overline{z}=\dfrac{1}{z}$ , 則

$$\int_C z^2\sin\overline{z}\,dz=\int_C z^2\sin\frac{1}{z}\,dz$$

令 $f(z)=z^2\sin\dfrac{1}{z}$, 故 $f(z)$ 在 $C$ 內具有 $z=0$ 的本性奇點 , 且 $f(z)$ 對 $z=0$ 展開的 Laurent's 級數爲

$$f(z)=z^2\,\sin\frac{1}{z}=z^2\left(\frac{1}{z}-\frac{1}{3!}\frac{1}{z^3}+\cdots\right)$$

故 Res $f(0)=-\dfrac{1}{6}$ , 則

$$\int_C z^2\,\sin\overline{z}\,dz=\int_C z^2\,\sin\frac{1}{z}\,dz=2\pi i\,\text{Res } f(0)=2\pi i\left(-\frac{1}{6}\right)=-\frac{\pi i}{3}$$

**27.** Evaluate $\oint_C z^2\exp\left(\dfrac{2}{z}\right)dz=?\quad C:|z|=2.$ 《中央電機》

《解》☞ 令 $f(z)=z^2\exp\left(\dfrac{2}{z}\right)$ , 故在 $z=0$ 處爲本性奇異點 , 又

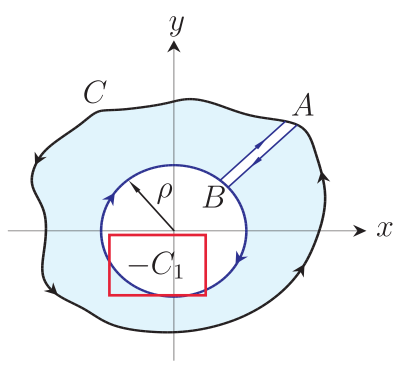
$$\exp\left(\frac{2}{z}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2}{z}\right)^n=1+\left(\frac{2}{z}\right)+\frac{1}{2!}\left(\frac{2}{z}\right)^2+\frac{1}{3!}\left(\frac{2}{z}\right)^3+\cdots$$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \cos\theta\,(2r^3)\Big|_{r=0}^{r=1} d\theta \\ &= \int_0^{\frac{\pi}{2}} 2\,\cos\theta\,d\theta = 2\sin\theta\Big|_0^{\frac{\pi}{2}} = 2 \end{aligned}$$

**14.** Let $\phi(x\,,\,y) = x + \ln(x^2 + y^2)$ and $\vec{u}(x\,,\,y) = (x^2\cos y)\vec{i} + (y^2\sin x)\vec{j}$ be 2-D scalar and vector functions, respectively; and let

$$\vec{f}(x\,,\,y) = \nabla\phi + \nabla\times\vec{u}$$

Evaluate the line integral $\oint_C \vec{f}\cdot\vec{n}\,ds$ , where $C$ is any simple closed curve enclosing the point $(0\,,\,0)$ and $\vec{n}$ denoting a unit vector outward normal to the curve $C$. 《台大機械》




《解》 ☞ 因

$$\begin{aligned} \nabla\cdot\vec{f} &= \nabla^2\phi + \nabla\cdot(\nabla\times\vec{u}) \\ &= -\frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} + 0 \\ &= 0 \end{aligned}$$

設 $C + AB + (-C_1) + BA$ 所圍的區域為 $R$ (如圖) , 故

$$\oint_{C+AB+(-C_1)+BA} \vec{f}\cdot\vec{n}\,ds = \iint_R (\nabla\cdot\vec{f}\,)\,dxdy = 0$$

故

$$\oint_C \vec{f}\cdot\vec{n}\,ds = \oint_{C_1} \vec{f}\cdot\vec{n}\,ds \tag{1}$$

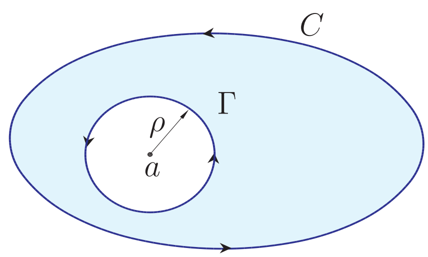
**10.** 証明 $f(z)$ 在簡單的封閉曲線 $C$ 上及其內部爲可解析函數，且 $a$ 爲 $C$ 內任意一點，則

$$f(a)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-a}dz$$

其中 $C$ 爲逆時針方向。同時 $f(z)$ 在 $z=a$ 處的 $n$ 階導數爲

$$f^{(n)}(a)=\frac{n!}{2\pi i}\oint_C\frac{f(z)}{(z-a)^{n+1}}dz\quad(n=1\,,\,2\,,\,\cdots)$$

《中正機械、成大電機》




**《解》** ☞ 令 $\Gamma$ 爲 $C$ 中以 $a$ 爲圓心，半徑爲 $\rho$ $(\rho\to 0)$ 的圓 (如圖)。則 $\Gamma$ : $|z-a|=\rho$，則 $z-a=\rho\, e^{i\theta}$、$\theta=0\sim 2\pi$、$dz=i\rho\, e^{i\theta}d\theta$，故

(1) $n=0$

因 $\dfrac{f(z)}{(z-a)}$ 在 $C$ 中，除了 $z=a$ 外均可解析，則由 Cauchy's 定理可知

$$\begin{aligned}\oint_C\frac{f(z)}{z-a}dz &= \lim_{\rho\to 0}\oint_\Gamma\frac{f(z)}{z-a}dz\\ &= \lim_{\rho\to 0}\int_0^{2\pi}\frac{f(a+\rho\, e^{i\theta})\,i\,\rho\, e^{i\theta}}{\rho e^{i\theta}}d\theta\\ &= \lim_{\rho\to 0} i\int_0^{2\pi} f(a+\rho\, e^{i\theta})d\theta\\ &= i\int_0^{2\pi}\lim_{\rho\to 0} f(a+\rho\, e^{i\theta})\,d\theta\\ &= i\int_0^{2\pi} f(a)\,d\theta=2\pi i f(a)\end{aligned}$$

故

$$f(a)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-a}dz \tag{1}$$

(2) 將 (1) 式中的 $a$ 視爲變數，對 (1) 式兩端取 $a$ 的微分可得

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right] dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\, dz$$

同理

$$f''(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{(z-a)^2}\right] dz = \frac{1}{2\pi i}\oint_C \frac{2\, f(z)}{(z-a)^3}\, dz$$

$$f'''(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{2f(z)}{(z-a)^3}\right] dz = \frac{1}{2\pi i}\oint_C \frac{3!\, f(z)}{(z-a)^4}\, dz$$

故

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad (n = 0\,,\,1\,,\,2\,,\,\cdots)$$

**11.** Evaluate (a) $\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\, dz$ (b) $\oint_C \frac{e^{2z}}{(z+1)^4}\, dz$

where $C$ is the $|z| = 3$. 《清華動機》

**《解》** ☞

(a) 因 $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$，故

$$\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\, dz = \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}\, dz - \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}\, dz$$

令 $f(z) = \sin(\pi z^2)+\cos(\pi z^2)$, 由 Cauchy's 積分公式，可得

$$\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}\, dz = 2\pi i\, f(2) = 2\pi i\{\sin(\pi\cdot 2^2)+\cos(\pi\cdot 2^2)\} = 2\pi i$$

$$\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}\, dz = 2\pi i\, f(1) = 2\pi i\{\sin(\pi\cdot 1^2)+\cos(\pi\cdot 1^2)\} = -2\pi i$$

故

$$\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\, dz = 2\pi i - (-2\pi i) = 4\pi i$$

(b) 將 $\oint_C \frac{e^{2z}}{(z+1)^4}\, dz$，與 Cauchy 積分公式

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{[n]}(a)$$

**7.** Evaluate the following integrals $\displaystyle\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\,d\theta.$ 《中央光電》

**《解》** ☞ 令 $C: |z|=1$，故 $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$、$d\theta = \frac{dz}{iz}$

$$\begin{aligned}\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\,d\theta &= \text{Im}\int_0^{2\pi} \frac{i+e^{i\theta}}{3+\cos\theta}\,d\theta \\ &= \text{Im}\oint_C \frac{i+z}{3+\frac{1}{2}(z+\frac{1}{z})}\frac{dz}{iz} \\ &= \text{Im}\oint_C \frac{2(i+z)}{i(z^2+6z+1)}\,dz \qquad (1)\end{aligned}$$

令

$$f(z) = \frac{2(i+z)}{i(z^2+6z+1)}$$

故 $f(z)$ 具有 $z=-3\pm 2\sqrt{2}$ 的一階 poles, 只有 $z=-3+2\sqrt{2}$ 在 $C$ 內，且

$$\text{Res } f(-3+2\sqrt{2}) = \frac{2(i+z)}{i(2z+6)}\Big|_{z=-3+2\sqrt{2}} = \frac{2(i-3+2\sqrt{2})}{4\sqrt{2}\,i}$$

故

$$\begin{aligned}\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\,d\theta &= \text{Im}\oint_C \frac{2(i+z)}{i(z^2+6z+1)}\,dz \\ &= \text{Im}\,\{2\pi i\,\text{Res } f(-3+2\sqrt{2})\} \\ &= \text{Im}\{2\pi i\cdot\frac{2(i-3+2\sqrt{2})}{4\sqrt{2}\,i}\} \\ &= \frac{1}{\sqrt{2}}\pi\end{aligned}$$

**8.** Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta.$ 《交大機械》

# 13.7 特殊型式的積分

## 13.7.1 避點積分

設線積分爲 $\oint_C F(z)\,dz$，若 $F(z)$ 在 $C$ 上具有有限個奇異點，則積分時必須避開所有的奇異點才能積分, 此種方法稱爲避點積分。

1. 定理 1*：設 $F(x)$ 爲有理式，且 $F(z)$ 滿足

(1) 設 $F(z)$ 在上半平面具有有限個極點。

(2) $F(z)$ 在 $x$ 軸上具有有限個單極點 (simple pole)。

(3) $|F(z)| \leq \dfrac{M}{R^k}$，$\forall |z| \geq R$，而且 $k > 1$ 及 $M > 0$。

$$\int_{-\infty}^{\infty} F(x)\,dx = 2\pi i \sum_{\text{上半平面}} \text{Res}\ F(z) + \pi i \sum_{\text{實軸上}} \text{Res}\ F(z)$$

2. 定理 2： 設 $F(x)$ 爲有理式，且 $F(z)$ 滿足

(1) 設 $F(z)$ 在上半平面具有有限個極點。

(2) $F(z)$ 在 $x$ 軸上具有有限個單極點 (simple pole)。

(3) $|F(z)| \leq \dfrac{M}{R^k}$，$\forall |z| \geq R$，而且 $k > 0$ 及 $M > 0$。

同時 $m > 0$，則

$$\int_{-\infty}^{\infty} F(x)\,e^{imx}\,dx = 2\pi i \sum_{\text{上半平面}} \text{Res}\ \{F(z)\,e^{imz}\} + \pi i \sum_{\text{實軸上}} \text{Res}\ \{F(z)\,e^{imz}\}$$

$$\int_{-\infty}^{\infty} F(x)\,e^{-imx}\,dx = -\left\{2\pi i \sum_{\text{下半平面}} \text{Res}\ \{F(z)\,e^{-imz}\} + \pi i \sum_{\text{實軸上}} \text{Res}\ \{F(z)\,e^{-imz}\}\right\}$$

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} F(x)\ \cos mx\,dx = \text{Re}\left\{ \int_{-\infty}^{\infty} F(x)\,e^{imx}\,dx \right\} \\ \displaystyle\int_{-\infty}^{\infty} F(x)\ \sin mx\,dx = \text{Im}\left\{ \int_{-\infty}^{\infty} F(x)\,e^{imx}\,dx \right\} \end{cases}$$

*本定理讀者請參考 Erwin Kreyszig, *Advanced Engineering Mathematics*, 8th Edition, pp.793, John Wiley & Sons, Inc., 1999.

$$\begin{aligned} &= -ie^{-t}\int_0^\infty \frac{e^{-tr}}{r^{\frac{1}{2}}}dr \quad (\text{令 } u=tr\,,\, du=tdr\,) \\ &= -ie^{-t}\int_0^\infty \frac{t^{\frac{1}{2}}e^{-u}}{u^{\frac{1}{2}}}\frac{du}{t} = -i\frac{e^{-t}}{t^{\frac{1}{2}}}\int_0^\infty u^{\frac{1}{2}-1}e^{-u}du \\ &= -i\frac{e^{-t}}{t}\Gamma(\frac{1}{2}) = -i\frac{\sqrt{\pi}e^{-t}}{t^{\frac{1}{2}}} \end{aligned}$$

(d) 在 $C_\rho$ 上：$s=-1+\rho e^{i\theta}$，故 $ds=\rho ie^{i\theta}d\theta$ 且 $\theta:\pi\to-\pi$

$$\begin{aligned} \lim_{\rho\to0}\left|\int_{C_\rho}\frac{e^{st}}{\sqrt{s+1}}ds\right| &= \lim_{\rho\to0}\left|\int_\pi^{-\pi}\frac{e^{t(-1+\rho e^{i\theta})}}{(\rho e^{i\theta})^{1/2}}\rho ie^{i\theta}d\theta\right| \\ &\le \lim_{\rho\to0}\int_\pi^{-\pi}\left|e^{t(-1+\rho\cos\theta+i\rho\sin\theta)}\rho^{\frac{1}{2}}ie^{i\frac{\theta}{2}}d\theta\right| \\ &\le \lim_{\rho\to0}(e^{-t+t\rho\cos\theta}\rho^{\frac{1}{2}}\cdot2\pi)=0 \end{aligned}$$

故

$$\lim_{\rho\to0}\int_{C_\rho}\frac{e^{st}}{\sqrt{s+1}}ds=0$$

(e) 在 GF 上：$S=-1+re^{-\pi i}$，故 $ds=e^{-\pi i}dr$ 且 $r:\rho\to(R-1)$

$$\begin{aligned} \lim_{\substack{R\to\infty\\ \rho\to0}}\int_{GE}F(s)e^{st}ds &= \lim_{\substack{R\to\infty\\ \rho\to0}}\int_\rho^{R-1}\frac{e^{t(-1-r)}}{(re^{-\pi i})^{\frac{1}{2}}}e^{-\pi i}dr \\ &= e^{-i\frac{\pi}{2}}\int_0^\infty\frac{e^{-t}e^{-tr}}{r^{\frac{1}{2}}}dr = -ie^{-t}\int_0^\infty\frac{e^{-tr}}{r^{\frac{1}{2}}}dr \\ &= -i\frac{\sqrt{\pi}e^{-t}}{t^{\frac{1}{2}}} \end{aligned}$$

(f) 把 (a)、(b)、(c)、(d)、(f) 代入 (1) 式可得

$$\frac{1}{2\pi i}\left[\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{st}}{\sqrt{s+1}}ds+0-\frac{i\sqrt{\pi}e^{-t}}{t^{\frac{1}{2}}}+0-\frac{i\sqrt{\pi}e^{-t}}{t^{\frac{1}{2}}}\right]=0$$

故

$$\mathscr{L}^{-1}\{\frac{1}{\sqrt{s+1}}\}=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{st}}{\sqrt{s+1}}ds=\frac{1}{2\pi i}\cdot2i\cdot\frac{\sqrt{\pi}e^{-t}}{t^{\frac{1}{2}}}=\frac{e^{-t}}{\sqrt{\pi t}}$$

(g) 另解：$\mathscr{L}^{-1}\{\frac{1}{\sqrt{s+1}}\}=e^{-t}\mathscr{L}^{-1}\{\frac{1}{\sqrt{s}}\}=e^{-t}\frac{t^{-1/2}}{\Gamma(\frac{1}{2})}=\frac{e^{-t}}{\sqrt{\pi t}}$

**16.** 試證 $\int_{-\infty}^{\infty}e^{-x^2}\cos 2bx\,dx=\sqrt{\pi}e^{-b^2}$。 《台科大電子》

故

$$\int_0^\infty \frac{\ln x}{\sqrt{x}(a^2+x^2)}dx = -\text{Im}\left\{\frac{\pi}{a^{\frac{3}{2}}}(\ln a + \frac{\pi}{2}i)e^{-\frac{\pi}{4}i}\right\} = \frac{\pi}{a^{\frac{3}{2}}}(\frac{\ln a}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}})$$

**23.** 設 $f(x)$ 滿足

(1) $f(z)$ 在整個複數平面上，除了有限個極點外，均爲可解析函數。

(2) $f(z)$ 在正實軸上 (包含原點) 無奇點。

(3) 當 $z \to 0$ 及 $z \to \infty$ 時 $|z^{1+\alpha}f(z)|$ 均勻收斂到零，其中 $\alpha$ 爲任意正實數。

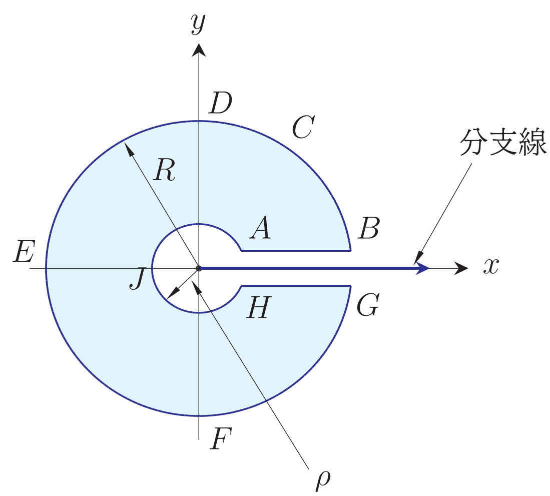
試証

$$\int_0^\infty f(x)\,dx = -\sum_{\text{all poles}} \text{Res}\,\{f(z)\ln z\}$$

$$\int_0^\infty f(x)\ln x\,dx = -\frac{1}{2}\text{Re}\sum_{\text{all poles}} \text{Res}\,\{f(z)(\ln z)^2\}$$

$$\int_0^\infty f(x)\,dx = -\frac{1}{2\pi}\text{Im}\sum_{\text{all poles}} \text{Res}\,\{f(z)(\ln z)^2\}$$

其中 $0 < arg(z) < 2\pi$。




**《解》** ☞

(a) 考慮一圍線 $C$ (如圖) 的線積分

$$\lim_{\substack{R\to\infty\\ \rho\to 0}} \oint_C f(z)\ln z\,dz = \lim_{\substack{R\to\infty\\ \rho\to 0}} \left[\int_{BEG} f(z)\ln z\,dz + \int_{GH} f(z)\ln z\,dz + \int_{HJA} f(z)\ln z\,dz + \int_{AB} f(z)\ln z\,dz\right] \quad (1)$$