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序

數學是所有科學之母, 洩漏天機的語言，微積分這一門學問, 它是僅次於歐氏幾何為數學上最偉大的成就之一, 也是近代所有理工商等學門的基礎學科，因此是研讀理工商的同學必修的一門課程，但是因為它的理論艱澀繁瑣, 枯燥難學，所以是很多莘莘學子在學習上的一大夢魘。近年來由於數位科技的發達和網路的普及化下，學校在雲端，教室在網路，沒有上下課的鐘聲，上課的黑板永遠不會被擦掉，忘掉了可隨時回頭再複習，只要上網就可上課，進度可自行控制，同學是自己學習課程的主人，翻轉教室 (flipped classroom)、MOOC (Massive Open Online Course)、個人學習 (personalized learning) 現在已是整個世界學習的主流趨勢，因此喻超凡老師秉著服務同學的初衷，參考國內外著名之微積分叢書，以及在國立大學及全國各大補習班任教的教學心得，提綱挈領的將重點及觀念，以結構化的方式放在每一章節的開始，並於每一章節的精選範例中, 加入重要的題型做整體而詳細的思路分析和講解說明，精編細撰出這本"翻轉微積分"，期能幫助想藉由翻轉學習的同學，在短時間內能對微積分有全盤性的認識及了解，進而讓同學翻轉成績, 翻轉未來。

本書手稿雖經多次修訂及校對, 但仍難免有所疏漏之處, 敬請各位老師和同學不吝賜教，網址為 http://www.superyu.idv.tw。

喻超凡雲端教室

喻超凡

2015. 6.

目錄

第 4 章 定積分及其應用

4.1 定積分 (The Definite Integral)

4.1.1 定積分的基本性質

1. 定義

(1) $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i)\ \Delta x_i$, 其中 $z_i \in [x_{i-1}\ ,\ x_i]$, $\Delta x_i = x_i - x_{i-1}$ 。

(2) 定積分 $\int_a^b f(x)\,dx$ 的幾何意義 , 爲 $f(x)$ 的圖形在直線 $x = a$ 及 $x = b$ 之間與 x 軸所圍的有向面積 , 如圖 (4.1) 。

(a) 在 x 軸上方的面積爲正 。

(b) 在 x 軸下方的面積爲負 。

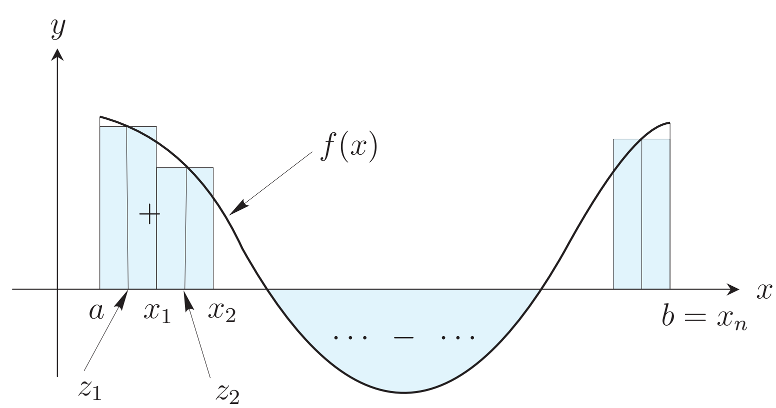

Figure 4.1: 定積分

2. 基本性質

(1) $\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$

(2) $\displaystyle\int_a^a f(x)\ dx = 0$

故 $\displaystyle\int_a^b f(x)\ dx + \int_b^a f(x)\ dx = 0$, 則 $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

(3) $\displaystyle\int_a^b f(x)\ dx = \int_a^b f(t)\ dt = \int_a^b f(u)\ du$

(4) $\displaystyle\int_a^b \{f(x) \pm g(x)\}\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

(5) $\displaystyle\int_a^b k\, f(x)\ dx = k \int_a^b f(x)\ dx$; k 為常數。

(6) $f(x) \leq g(x)$ 且 $(a < b)$, 則 $\displaystyle\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$

(7) $\displaystyle\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx$

4.1.2 積分均值定理

1. 設 $f(x)$ 在 $x \in [a\,,\,b]$ 中為連續 , 則 $\exists c \in (a\,,\,b)$, 使得

$$\int_a^b f(x)\ dx = f(c)\,(b-a)$$

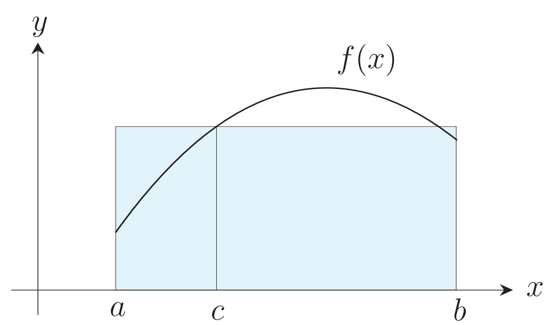

Figure 4.2: 積分均值定理

2. 等價的定義

(1) $\displaystyle\int_a^{a+h} f(x)\ dx = f(a+\theta h)\cdot h$; $(0 < \theta < 1)$

(2) $\displaystyle\int_a^b f(x)\ dx = f(c)\int_a^b 1\,dx$; $c \in (a\,,\,b)$

(3) $\displaystyle\int_a^b f(x)g(x)\ dx = f(c)\int_a^b g(x)\ dx$; $c \in (a\,,\,b)$, 且 $g : [a,b] \to \mathbb{R}$ 不改變符號 。

4.1.3 微積分基本定理

1. 微積分基本定理 (Fundamental Theorem of the Calculus)

設 f 在 $[a\,,\,b]$ 上爲連續

(1) 若 $F(x) = \displaystyle\int_a^x f(t)\,dt$, $\forall\, x \in [a\,,\,b\,]$, 則 $F'(x) = f(x)$, $\forall\, x \in (a\,,\,b)$

(2) 若 $G(x)$ 爲 $f(x)$ 之反導數 , 則

$$\int_a^b f(x)\ dx = G(b) - G(a) = G(x)\Big|_a^b$$

2. 微積分基本定理推廣

(1) 若 $F(x) = \displaystyle\int_a^{u(x)} f(t)\ dt$, 則

$$F'(x) = \frac{d}{dx}\int_a^{u(x)} f(t)\ dt = \frac{d}{du}\{\int_a^u f(t)\ dt\}\cdot\frac{du}{dx} = f(u(x))\,u'(x)$$

(2) 若 $F(x) = \displaystyle\int_{u(x)}^{v(x)} f(t)\ dt$, 則

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\ dt \\ &= \frac{d}{dx}\Big\{\int_a^{v(x)} f(t)\ dt - \int_a^{u(x)} f(t)\ dt\Big\} \\ &= f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x) \end{aligned}$$

精選範例

1. 何謂積分均值定理 (Mean Value Theorem for definite integrals)？並證明之。

《解》☞

(1) 積分均值定理：設 $f(x)$ 在 $x \in [a\,,\,b]$ 中為連續，則 $\exists c \in (a\,,\,b)$，使得

$$\int_a^b f(x)\;dx = f(c)\,(b-a)$$

(2) 因為 $f(x)$ 在 $[a\,,\,b]$ 為連續，由極值定理可知

$$\exists\, u\,,\, v \in [a\,,\,b]\,,\ni f(u) \le f(x) \le f(v)\,,\forall x \in (a\,,\,b)$$

故

$$\int_a^b f(u)\;dx \le \int_a^b f(x)\;dx \le \int_a^b f(v)\;dx$$

即

$$f(u)(b-a) \le \int_a^b f(x)\;dx \le f(v)(b-a)$$

上式同除 $(b-a)$，可得

$$f(u) \le \frac{1}{b-a}\int_a^b f(x)\;dx \le f(v)$$

再由介值定理可知，$\exists c \in (u\,,\,v) \subset (a\,,\,b)$，使得

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

故

$$\int_a^b f(x)\;dx = f(c)\,(b-a) \quad ; \quad c \in (a\,,\,b)$$

2. 何謂微積分基本定理 (The Fundamental Theorem of Calculus)？並證明之。

《解》 ☞

(1) 微積分基本定理：設 $f(x)$ 在 $[a\,,\,b]$ 中爲連續

(a) 若 $F(x)=\displaystyle\int_a^x f(t)dt\ ,\ \forall\, x\in[a\,,\,b]$，則 $F'(x)=f(x)\ ,\ \forall\, x\in(a\,,\,b)$

(b) 若 $G(x)$ 爲 $f(x)$ 之反導數，則

$$\int_a^b f(x)dx=G(b)-G(a)=G(x)\Big|_a^b$$

(2)

(a)

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \quad (\text{導數的定義得知})\\
&= \lim_{h\to 0}\frac{\displaystyle\int_a^{x+h} f(t)dt-\int_a^x f(t)dt}{h} \quad (F(x)\text{ 的定義得知})\\
&= \lim_{h\to 0}\frac{\displaystyle\int_a^{x} f(t)dt+\int_x^{x+h} f(t)dt-\int_a^x f(t)dt}{h} \quad (\text{定積分的基本性質})\\
&= \lim_{h\to 0}\frac{\displaystyle\int_x^{x+h} f(t)dt}{h}\\
&= \lim_{h\to 0}\frac{f(x+\theta h)\cdot h}{h}\ ;\ \ (0<\theta<1)\quad (\text{積分均值定理})\\
&= \lim_{h\to 0} f(x+\theta h)\\
&= f(\lim_{h\to 0}(x+\theta h)) \quad (\text{連續性質})\\
&= f(x)
\end{aligned}$$

(b) 因 $F'(x)=f(x)$、$G'(x)=f(x)$，故

$$F(x)=G(x)+k \tag{1}$$

令 $x=a$ 代回 (1) 式可得

$$F(a)=G(a)+k$$

即

$$F(a)=\int_a^a f(t)dt=0=G(a)+k$$

故 $k=-G(a)$, 再令 $x=b$ 代回 (1) 式可得

$$F(b)=\int_a^b f(t)dt=G(b)+k=G(b)-G(a)=G(x)\Big|_a^b$$

3. 求極限值 $\lim\limits_{x\to 0}\dfrac{\displaystyle\int_0^{x^2}\sin(t^2)\ dt}{x^6}$ 《台大》

《提示》 ➥ $\dfrac{0}{0}$ 不定型

《解》 ☞

$$\begin{aligned}\lim_{x\to 0}\frac{\displaystyle\int_0^{x^2}\sin(t^2)\ dt}{x^6} &= \lim_{x\to 0}\frac{\sin x^4\cdot 2x}{6x^5}=\lim_{x\to 0}\frac{\sin x^4}{3x^4}\\ &= \lim_{x\to 0}\frac{\cos x^4\cdot 4x^3}{12x^3}=\lim_{x\to 0}\frac{\cos x^4}{3}\\ &= \frac{1}{3}\end{aligned}$$

4. Find $\lim\limits_{x\to +\infty}\dfrac{\displaystyle\int_0^{x}\exp(t^2)\ dt}{x^2}$. 《成大》

《提示》 ➥ $\dfrac{\infty}{\infty}$ 不定型

《解》 ☞ $\lim\limits_{x\to +\infty}\dfrac{\displaystyle\int_0^{x}\exp(t^2)\ dt}{x^2}=\lim\limits_{x\to +\infty}\dfrac{e^{x^2}}{2x}=\lim\limits_{x\to +\infty}\dfrac{2xe^{x^2}}{2}=+\infty$

5. 求 $\lim_{x\to\infty} x\int_0^x e^{(t^2-x^2)}\ dt$ 。

《解》 ☞

$$\begin{aligned}
\lim_{x\to\infty} x\int_0^x e^{(t^2-x^2)}\ dt &= \lim_{x\to\infty} \frac{x\int_0^x e^{t^2}\ dt}{e^{x^2}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x\int_0^x e^{t^2} dt)}{\frac{d}{dx}e^{x^2}} \\
&= \lim_{x\to\infty} \frac{\int_0^x e^{t^2}\ dt + xe^{x^2}}{2xe^{x^2}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\int_0^x e^{t^2}\ dt + xe^{x^2})}{\frac{d}{dx}(2xe^{x^2})} \\
&= \lim_{x\to\infty} \frac{e^{x^2}+e^{x^2}+2x^2e^{x^2}}{2e^{x^2}+4x^2e^{x^2}} = \lim_{x\to\infty} \frac{1+x^2}{1+2x^2} \\
&= \lim_{x\to\infty} \frac{2x}{4x} = \frac{1}{2}
\end{aligned}$$

6. 設函數 $f(x)$ 滿足 $f(x) = 1+3x^2+4x\int_0^1 f(x)\ dx$，求 $f(x) = ?$

《解》 ☞ 設 $\int_0^1 f(x)\ dx = A$，故

$$A = \int_0^1 f(x)\ dx = \int_0^1 (1+3x^2+4x\cdot A)\ dx = (x+x^3+2x^2\cdot A)\Big|_0^1 = 2+2A$$

則 $A=-2$，故

$$f(x) = 1+3x^2+4x\cdot(-2) = 1-8x+3x^2$$

7. 求下列之值

(a) $\dfrac{d}{dx}\displaystyle\int_1^{2x} \frac{\sin t}{t^2}\ dt$ (b) $\dfrac{d}{dx}\displaystyle\int_{\sin^2 x}^{\cos^2 x} \sqrt{t}\ dt$ 《輔大》

《解》☞

(a) $\dfrac{d}{dx}\displaystyle\int_1^{2x}\frac{\sin t}{t^2}\,dt=\frac{\sin(2x)}{(2x)^2}\cdot\frac{d}{dx}(2x)=\frac{\sin 2x}{(2x)^2}\cdot 2=\frac{\sin 2x}{2x^2}$

(b)

$$\begin{aligned}\frac{d}{dx}\int_{\sin^2 x}^{\cos^2 x}\sqrt{t}\,dt &= \frac{d}{dx}\Big\{\int_a^{\cos^2 x}\sqrt{t}\,dt-\int_a^{\sin^2 x}\sqrt{t}\,dt\Big\}\\ &= \sqrt{\cos^2 x}\,\frac{d}{dx}(\cos^2 x)-\sqrt{\sin^2 x}\,\frac{d}{dx}(\sin^2 x)\\ &= |\cos x|\cdot 2\cos x\cdot(-\sin x)-|\sin x|\cdot 2\sin x\cdot\cos x\\ &= -\sin 2x\cdot(|\cos x|+|\sin x|)\end{aligned}$$

8. 已知 $f(x)$ 爲連續函數，且 $\displaystyle\int_0^x f(t)\,dt=3+x+k\cos 2x$，試求

(a) k (b) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)\,dx$ (c) $f'(\dfrac{\pi}{2})$

《解》☞

(a) 因

$$\int_0^x f(t)\,dt=3+x+k\cos 2x$$

令 $x=0$ 代回上式可得

$$\int_0^0 f(t)\,dt=3+0+k=0\quad(\text{因 } f(x) \text{ 連續函數})$$

故 $k=-3$。

(b) 因

$$\int_0^x f(t)dt=3+x-3\cos 2x$$

對上式兩端微分

$$\frac{d}{dx}\int_0^x f(t)dt=\frac{d}{dx}(3+x-3\cos 2x)$$

可得

$$f(x)=1+6\sin 2x$$

故

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)\ dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+6\sin 2x)dx = (x-3\cos 2x)\ \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4}+3$$

(c) 因 $f(x)=1+6\sin 2x$，故 $f'(x)=12\cos 2x$，則 $f'(\frac{\pi}{2})=-12$ 。

9. 求 $\lim\limits_{a\to 0}\int_a^{2a} \frac{e^{2x}}{x}\ dx$

《提示》 ➥ 積分均值定理僅適用於連續函數 。

《解》 ☞ 因 e^{2x} 爲連續函數，故 $\exists\, c\in(a\,,\,2a)$，使得

$$\begin{aligned}\int_a^{2a} \frac{e^{2x}}{x}\ dx &= e^{2c}\int_a^{2a}\frac{1}{x}\ dx = e^{2c}\ \ln|x|\ \Big|_a^{2a}\\ &= e^{2c}\left(\ln|2a|-\ln|a|\right) = e^{2c}\ \ln|\frac{2a}{a}|\\ &= e^{2c}\ \ln 2\end{aligned}$$

因 $a<c<2a$，故 $a\to 0$ 時 $c\to 0$，則

$$\lim_{a\to 0}\int_a^{2a}\frac{e^{2x}}{x}dx = \lim_{a\to 0} e^{2c}\ \ln 2 = \lim_{c\to 0} e^{2c}\ \ln 2 = \ln 2$$

10. 已知 $f(x)$ 爲連續函數，試證

(a) 若$f(x)$ 爲偶函數，則$\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$

(b) 若$f(x)$ 爲奇函數，則$\int_{-a}^{a} f(x)\ dx = 0$

《提示》 ➥ $f(-x)=f(x) \Longleftrightarrow f(x)$ 爲偶函數

$f(-x)=-f(x) \Longleftrightarrow f(x)$ 爲奇函數

《解》☞

(a) 因 $f(x)$ 為偶函數，故 $f(-x)=f(x)$，則

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx+\int_{-a}^{0} f(x)\,dx \\
&\quad (\text{令 } u=-x\text{，故 } dx=-du\text{，} \begin{array}{c|c|c} x & -a & 0 \\ \hline u & a & 0 \end{array}) \\
&= \int_{0}^{a} f(x)\,dx-\int_{a}^{0} f(-u)\,du \\
&= \int_{0}^{a} f(x)\,dx+\int_{0}^{a} f(-x)\,dx \\
&= \int_{0}^{a} f(x)\,dx+\int_{0}^{a} f(x)\,dx \\
&= 2\int_{0}^{a} f(x)dx
\end{aligned}$$

(b) 因 $f(x)$ 為奇函數，故 $f(-x)=-f(x)$，則

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx+\int_{-a}^{0} f(x)\,dx \\
&\quad (\text{令 } u=-x\text{，故 } dx=-du\text{，} \begin{array}{c|c|c} x & -a & 0 \\ \hline u & a & 0 \end{array}) \\
&= \int_{0}^{a} f(x)\,dx-\int_{a}^{0} f(-u)\,du \\
&= \int_{0}^{a} f(x)\,dx+\int_{0}^{a} f(-x)\,dx \\
&= \int_{0}^{a} f(x)\,dx-\int_{0}^{a} f(x)\,dx \\
&= 0
\end{aligned}$$

11. 求下列積分

(a) $\displaystyle\int_{-2}^{2} \frac{\sin^3 x}{2+\cos x}dx$ 《淡大》 (b) $\displaystyle\int_{-7}^{7} x^3\sqrt{5x^6+3}dx$ 《淡大》

(c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 x}dx$

《提示》 ➥ 先判斷被積函數是否爲偶函數或奇函數。設 $f(x)$ 爲 $x \in [-\ell, \ell]$ 的函數

1. 常見的奇函數：x、x^3、x^5、$\cdots$；$\sin(ax)$、$\tan(ax)$；$\sin^{-1}(ax)$、$\tan^{-1}(ax)$；

   $\sinh(ax)$、$\tanh(ax)$

2. 常見的偶函數：常數、x^2、x^4、$\cdots$；$\cos(ax)$；$\cosh(ax)$；$|$奇偶函數$|$；$f($偶函數$)$

3. 奇偶函數的四則運算：

   偶函數 $\times(\div)$ 偶函數 $=$ 偶函數 ； 奇函數 $\times(\div)$ 奇函數 $=$ 偶函數

   偶函數 $\times(\div)$ 奇函數 $=$ 奇函數

   偶函數 $\pm$ 偶函數 $=$ 偶函數 ； 奇函數 $\pm$ 奇函數 $=$ 奇函數

   偶函數 $\pm$ 奇函數 $=$ 不奇不偶的函數

《解》 ☞

(a) 因 $\sin^3 x$ 爲奇函數，$2+\cos x$ 爲偶函數，故 $\dfrac{\sin^3 x}{2+\cos x}$ 爲奇函數，則

$$\begin{aligned}
\int_{-2}^{2} \frac{\sin^3 x}{2+\cos x}\,dx &= 1 \quad \int_{0}^{2} \frac{\sin^3 x}{2+\cos x}\,dx + \int_{-2}^{0} \frac{\sin^3 x}{2+\cos x}\,dx \\
& \quad (\text{令 } u=-x\text{，則 } dx=-du\ ) \\
&= \int_{0}^{2} \frac{\sin^3 x}{2+\cos x}\,dx - \int_{2}^{0} \frac{-\sin^3 u}{2+\cos u}\,du \\
&= \int_{0}^{2} \frac{\sin^3 x}{2+\cos x}\,dx - \int_{0}^{2} \frac{\sin^3 x}{2+\cos x}\,dx = 0
\end{aligned}$$

(b) 因 x^3 爲奇函數，$\sqrt{5x^6+3}$ 爲偶函數，故 $x^3\sqrt{5x^6+3}$ 爲奇函數，則

$$\begin{aligned}
\int_{-7}^{7} x^3\sqrt{5x^6+3}\,dx &= \int_{0}^{7} x^3\sqrt{5x^6+3}\,dx + \int_{-7}^{0} x^3\sqrt{5x^6+3}\,dx \\
& \quad (\text{令 } u=-x\text{，則 } dx=-du\ ) \\
&= \int_{0}^{7} x^3\sqrt{5x^6+3}\,dx - \int_{7}^{0} (-u^3)\sqrt{5u^6+3}\,du \\
&= \int_{0}^{7} x^3\sqrt{5x^6+3}\,dx - \int_{0}^{7} x^3\sqrt{5x^6+3}\,dx = 0
\end{aligned}$$

(c) 因 $\sqrt{1-\cos^2 x}$ 爲偶函數，則

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 x}\,dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\,dx \\
&= \int_{-\frac{\pi}{2}}^{0} (-\sin x)\,dx + \int_{0}^{\frac{\pi}{2}} \sin x\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\frac{\pi}{2}}^{0} (-\sin u) du + \int_{0}^{\frac{\pi}{2}} \sin x dx \\
&\quad (\text{令 } u = -x \text{，則 } dx = -du \text{)} \\
&= \int_{0}^{\frac{\pi}{2}} \sin x dx + \int_{0}^{\frac{\pi}{2}} \sin x dx \\
&= 2\int_{0}^{\frac{\pi}{2}} \sin x dx = 2(-\cos x)\Big|_{0}^{\frac{\pi}{2}} \\
&= 2
\end{aligned}
$$

12. 求下列積分

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^5 x\ dx$ 《清大》 (b) $\displaystyle\int_{-2}^{1} \frac{x}{\sqrt{3+x}}\ dx$

(c) $\displaystyle\int_0^{1} x^2(1-x)^{10}\ dx$ 《淡大》 (d) $\displaystyle\int_0^{1} x^2 e^x\ dx$ 《淡大》

《解》 ☞

(a)

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \cos^5 x\ dx &= \int_0^{\frac{\pi}{4}} \cos^4 x\ \cos x\ dx \\
&= \int_0^{\frac{\pi}{4}} (1-\sin^2 x)^2\ d(\sin x) \\
&= \int_0^{\frac{\pi}{4}} (1-2\sin^2 x+\sin^4 x)\, d(\sin x) \\
&= \{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\}\ \Big|_0^{\frac{\pi}{4}} \\
&= \frac{43\sqrt{2}}{120}
\end{aligned}
$$

(b) 令 $u=\sqrt{3+x}$，故 $u^2=3+x$，$2u\,du=dx$，且

x	-2	1
u	1	2

則

$$
\int_{-2}^{1} \frac{x}{\sqrt{3+x}}\ dx = \int_1^2 \frac{u^2-3}{u} 2u\ du = 2\int_1^2 (u^2-3)\ du
$$

$$= 2\{\frac{1}{3}u^3 - 3u\}\Big|_1^2 = -\frac{4}{3}$$

(c) 令 $u = 1 - x$, 故 $dx = -du$, 且

x	0	1
u	1	0

則

$$\begin{aligned}\int_0^1 x^2(1-x)^{10}\,dx &= \int_1^0 (1-u)^2u^{10}\,(-du)\\ &= \int_0^1 (u^{10} - 2u^{11} + u^{12})\,du\\ &= \{\frac{u^{11}}{11} - \frac{2}{12}u^{12} + \frac{u^{13}}{13}\}\Big|_0^1\\ &= \frac{1}{11} - \frac{1}{6} + \frac{1}{13} = \frac{1}{858}\end{aligned}$$

(d)

微	積
$+x^2$	e^x
$-2x$	e^x
$+2$	e^x
-0	e^x

$$\int_0^1 x^2e^x\,dx = \{x^2\,e^x - 2x\,e^x + 2e^x\}\Big|_0^1 = e - 2e + 2e - 2 = e - 2$$

13. 求下列積分

(a) $\displaystyle\int_{-1}^{2} |x^2 - x|\,dx$ 《淡大》 (b) $\displaystyle\int_0^2 \min\{3x\,,\,4 - x^2\}\,dx$

(c) $\displaystyle\int_2^7 x\left[\frac{x}{3}\right]dx$

《提示》 ➥ 先劃分積分區間, 以消除特殊函數。

《解》☞

(a) 令 $x^2-x=0$，故 $x(x-1)=0$，則 $x=0$、1

$$\begin{aligned}\int_{-1}^{2}|x^2-x|\ dx &= \int_{-1}^{0}(x^2-x)\ dx+\int_{0}^{1}-(x^2-x)dx+\int_{1}^{2}(x^2-x)dx\\ &= \{\frac{x^3}{3}-\frac{x^2}{2}\}\Big|_{-1}^{0}+\{-\frac{x^3}{3}+\frac{x^2}{2}\}\Big|_{0}^{1}+\{\frac{x^3}{3}-\frac{x^2}{2}\}\Big|_{1}^{2}\\ &= 0-(\frac{-1}{3}-\frac{1}{2})+(-\frac{1}{3}+\frac{1}{2})-0+(\frac{8}{3}-2)-(\frac{1}{3}-\frac{1}{2})\\ &= \frac{11}{6}\end{aligned}$$

(b) 令 $3x<4-x^2$，故 $x^2+3x-4<0$，即 $(x+4)(x-1)<0$，則 $-4<x<1$

$$\begin{aligned}\int_{0}^{2}\min\{3x\,,\,4-x^2\}\ dx &= \int_{0}^{1}\min\{3x\,,\,4-x^2\}\ dx+\int_{1}^{2}\min\{3x\,,\,4-x^2\}\ dx\\ &= \int_{0}^{1}3x\ dx+\int_{1}^{2}(4-x^2)\ dx\\ &= \frac{3}{2}x^2\Big|_{0}^{1}+(4x-\frac{x^3}{3})\Big|_{1}^{2}\\ &= \frac{3}{2}-0+(8-\frac{8}{3})-(4-\frac{1}{3})\\ &= \frac{19}{6}\end{aligned}$$

(c) 令 $\left[\frac{x}{3}\right]=n$，故 $n\le\frac{x}{3}<n+1$，則 $3n\le x<3(n+1)$，因此

$$\begin{aligned}\int_{2}^{7}x\left[\frac{x}{3}\right]\ dx &= \int_{2}^{3}(x\cdot 0)\ dx+\int_{3}^{6}(x\cdot 1)\ dx+\int_{6}^{7}(x\cdot 2)\ dx\\ &= 0+\frac{x^2}{2}\Big|_{3}^{6}+x^2\Big|_{6}^{7}\\ &= 18-\frac{9}{2}+49-36=\frac{53}{2}\end{aligned}$$

14. (a) 試證 $\displaystyle\int_{0}^{\frac{\pi}{2}} f(\sin x)\ dx=\int_{0}^{\frac{\pi}{2}} f(\cos x)\ dx$

(b) 利用 (a) 求 $\displaystyle\int_{0}^{\frac{\pi}{2}}\frac{\sin^n x}{\sin^n x+\cos^n x}\ dx$ 《中山材光》

《提示》➥ 利用互餘變換（令 $u=\frac{\pi}{2}-x$）證明。

《解》 ☞

(a) 令 $u = \frac{\pi}{2} - x$，故 $dx = -du$，且 $\begin{array}{c|c|c} x & 0 & \frac{\pi}{2} \\ \hline u & \frac{\pi}{2} & 0 \end{array}$，則

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_{\frac{\pi}{2}}^{0} f\Big(\sin(\frac{\pi}{2} - u)\Big)(-du) = \int_0^{\frac{\pi}{2}} f(\cos u)du = \int_0^{\frac{\pi}{2}} f(\cos x)dx$$

(b) 令

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}dx \tag{1}$$

由 (a) 可知

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}dx \tag{2}$$

由 (1) 式 + (2) 式可得

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\, dx \\ &= \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2} \end{aligned}$$

故

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx = \frac{\pi}{4}$$

15. (a) 試證 $\int_0^{\pi} xf(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)\, dx$

(b) 利用 (a) 求 $\int_0^{\pi} x\sin^2 x\, dx$ 《中興資工、土木》

《提示》 ➥ 利用互補變換（令 $u = \pi - x$）證明。

《解》 ☞

(a)

(i) 令 $u=\pi-x$, 故 $dx=-du$, 且 $\begin{array}{c|c|c} x & 0 & \pi \\ \hline u & \pi & 0 \end{array}$, 則

$$\begin{aligned} \int_0^{\pi} xf(\sin x)\ dx &= \int_{\pi}^{0} (\pi-u)f(\sin(\pi-u))(-du) \\ &= \int_0^{\pi} (\pi-u)f(\sin u)\ du \\ &= \pi\int_0^{\pi} f(\sin u)\ du - \int_0^{\pi} uf(\sin u)\ du \\ &= \pi\int_0^{\pi} f(\sin x)\ dx - \int_0^{\pi} xf(\sin x)\ dx \end{aligned}$$

將 $\int_0^{\pi} xf(\sin x)\ dx$ 移項到等號的左邊, 可得

$$2\int_0^{\pi} xf(\sin x)\ dx = \pi\int_0^{\pi} f(\sin x)\ dx$$

故

$$\int_0^{\pi} xf(\sin x)\ dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\ dx$$

(ii) 令 $u=\pi-x$, 故 $dx=-du$, 且 $\begin{array}{c|c|c} x & \frac{\pi}{2} & \pi \\ \hline u & \frac{\pi}{2} & 0 \end{array}$, 則

$$\begin{aligned} \int_0^{\pi} f(\sin x)dx &= \int_0^{\frac{\pi}{2}} f(\sin x)\ dx + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\ dx \\ &= \int_0^{\frac{\pi}{2}} f(\sin x)\ dx + \int_{\frac{\pi}{2}}^{0} f(\sin(\pi-u))(-du) \\ &= \int_0^{\frac{\pi}{2}} f(\sin x)\ dx + \int_0^{\frac{\pi}{2}} f(\sin u)\ du \\ &= \int_0^{\frac{\pi}{2}} f(\sin x)dx + \int_0^{\frac{\pi}{2}} f(\sin x)dx \\ &= 2\int_0^{\frac{\pi}{2}} f(\sin x)dx \end{aligned}$$

故

$$\int_0^{\pi} xf(\sin x)\ dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\ dx = \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}} f(\sin x)\ dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)\ dx$$

(b)

$$\begin{aligned}\int_0^{\pi} x\sin^2 x\ dx &= \frac{\pi}{2}\int_0^{\pi}\sin^2 x\ dx = \pi\int_0^{\frac{\pi}{2}}\sin^2 x\ dx\\ &= \pi\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}\ dx = \frac{\pi}{2}(x-\sin x\cos x)\Big|_0^{\frac{\pi}{2}}\\ &= \frac{\pi^2}{4}\end{aligned}$$

16. 求 $\int_{-1}^{2}|x|\cdot[x]\ dx = ?$ 《淡大》

《**提示**》➥ 分段討論以消去特殊函數

《**解**》☞ 因

$$|x| = \begin{cases}-x & ; x<0\\ x & ; x\geq 0\end{cases} \quad 又 \quad [x] = \begin{cases}-1 & ; -1<x<0\\ 0 & ; 0\leq x<1\\ 1 & ; 1\leq x<2\end{cases}$$

故

$$|x|\cdot[x] = \begin{cases}x & ; -1<x<0\\ 0 & ; 0\leq x<1\\ x & ; 1\leq x<2\end{cases}$$

則

$$\begin{aligned}\int_{-1}^{2}|x|\cdot[x]\ dx &= \int_{-1}^{0}x\ dx+\int_0^1 0\ dx+\int_1^2 x\ dx\\ &= \frac{x^2}{2}\Big|_{-1}^{0}+\frac{x^2}{2}\Big|_1^2\\ &= 0-\frac{1}{2}+2-\frac{1}{2}=1\end{aligned}$$

17. 求下列積分

(a) $\int_4^9\frac{1}{x-\sqrt{x}}dx$ 《中原》 (b) $\int_1^2\frac{\sqrt{x^2-1}}{x}dx$ 《海洋》

(c) $\int_0^3\frac{1}{(1+x)\sqrt{x}}dx$ 《海洋》

《解》☞

(a) $\dfrac{1}{x-\sqrt{x}} = x^{-1}(1-x^{-\frac{1}{2}})^{-1}$, 故 $r=-1$, 則令 $x=t^2$, $dx=2t\ dt$, 且

x	4	9
t	2	3

故

$$\int_4^9 \frac{1}{x-\sqrt{x}}\ dx = \int_2^3 \frac{1}{t^2-t}\ 2t\ dt = 2\int_2^3 \frac{1}{t-1}\ dt = 2\ln|t-1|\Big|_2^3 = 2\ln 2$$

(b) $\dfrac{\sqrt{x^2-1}}{x} = x^{-1}(-1+x^2)^{\frac{1}{2}}$, 故 $\dfrac{m+1}{n}=0$, 則令 $x^2-1=t^2$, 故 $2x\ dx = 2t\ dt$ 、$x=\sqrt{t^2+1}$, 且

x	1	2
t	0	$\sqrt{3}$

則

$$\begin{aligned}\int_1^2 \frac{\sqrt{x^2-1}}{x}\ dx &= \int_0^{\sqrt{3}} \frac{t}{\sqrt{t^2+1}}\cdot\frac{t}{\sqrt{t^2+1}}\ dt = \int_0^{\sqrt{3}} \frac{t^2+1-1}{t^2+1}\ dt \\ &= \int_0^{\sqrt{3}} 1\ dt - \int_0^{\sqrt{3}} \frac{1}{t^2+1}\ dt = t\ \Big|_0^{\sqrt{3}} - \tan^{-1} t\ \Big|_0^{\sqrt{3}} \\ &= \sqrt{3}-0-(\frac{\pi}{3}-0) = \sqrt{3}-\frac{\pi}{3}\end{aligned}$$

(c) $\dfrac{1}{(1+x)\sqrt{x}} = x^{-\frac{1}{2}}(1+x)^{-1}$, 故 $r=-1$, 則令 $x=t^2$, 故 $dx=2t\ dt$, 且

x	0	3
t	0	$\sqrt{3}$

則

$$\begin{aligned}\int_0^3 \frac{1}{(1+x)\sqrt{x}}\ dx &= \int_0^{\sqrt{3}} \frac{1}{(1+t^2)t}\,2t\ dt \\ &= \int_0^{\sqrt{3}} \frac{2}{1+t^2}\ dt = 2\tan^{-1} t\ \Big|_0^{\sqrt{3}} \\ &= 2\,(\,\frac{\pi}{3}-0\,) = \frac{2}{3}\pi\end{aligned}$$

18. 設 $x=e^y+y-1$, 試求 $\displaystyle\int_0^e y\ dx$ 之值 。 《文化》

《解》 ☞ 因爲 $x = e^y + y - 1$, 故 $dx = (e^y + 1)dy$, 且

x	0	e
y	0	1

則

$$\begin{aligned}\int_0^e y\ dx &= \int_0^1 y\cdot(e^y+1)dy = \int_0^1 y\,e^y\ dy + \int_0^1 y\ dy \\ &= \int_0^1 y\ d(e^y) + \frac{1}{2}y^2\Big|_0^1 = (y\,e^y - e^y)\Big|_0^1 + \frac{1}{2} - 0 \\ &= e - e + 1 + \frac{1}{2} = \frac{3}{2}\end{aligned}$$

19. 試求 a 、b 使得函數 $f(x) = (ax+b)\,e^x$ 滿足 $f(x) = e^x - 1 + \int_0^x f(t)\ dt$ 。

《解》 ☞ 因

$$f(x) = e^x - 1 + \int_0^x f(t)\ dt \tag{1}$$

故

$$f(0) = 1 - 1 + \int_0^0 f(t)\ dt = 1 - 1 + 0 = 0$$

又 $f(x) = (ax+b)e^x$, 故 $f(0) = b = 0$, 再將 $f(x)$ 代回 (1) 式可得

$$(ax+b)e^x = e^x - 1 + \int_0^x (at+b)e^t\ dt$$

將 $b = 0$ 代入上式可得

$$\begin{aligned}ax\cdot e^x &= e^x - 1 + \int_0^x at\,e^t\,dt \\ &= e^x - 1 + a(te^t - e^t)\Big|_0^x \\ &= e^x - 1 + axe^x - ae^x + a\end{aligned}$$

整理可得

$$(a-1)(1-e^x) = 0$$

因 $(1-e^x) \neq 0$, 故 $a - 1 = 0$, 即 $a = 1$ 。

20. 試求 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\,dx$，其中 a、$b \neq 0$。 《成大》

《解》☞

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\,dx &= \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x\,(\,a^2\dfrac{\sin^2 x}{\cos^2 x} + b^2)}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2\tan^2 x + b^2}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{a^2\tan^2 x + b^2}\,d(\tan x) \\
&= \lim_{t\to\frac{\pi}{2}} \int_0^{t} \frac{1}{a^2\tan^2 x + b^2}\,d(\tan x) \\
&= \lim_{t\to\frac{\pi}{2}} \frac{1}{|ab|}\,\tan^{-1}(\frac{|a|}{|b|}\tan t) = \frac{\pi}{2|ab|}
\end{aligned}$$

21. Let $g(x)$ be the inverse function of strictly increasing function $f(x) = x^5 + 3x^3 + 1$. Then $\displaystyle\int_{f(0)}^{f(1)} g(x)\,dx = ?$ 《台大C》

《解》☞ 令 $y = f(x)$，故 $x = f^{-1}(y) = g(y)$，$dy = f'(x)dx$，且

y	$f(0)$	$f(1)$
x	0	1

則

$$\begin{aligned}
\int_{f(0)}^{f(1)} g(x)\,dx &= \int_{f(0)}^{f(1)} g(y)\,dy \\
&= \int_0^1 x\,f'(x)\,dx = \int_0^1 x(5x^4 + 9x^2)\,dx \\
&= \{\frac{5}{6}x^6 + \frac{9}{4}x^4\}\Big|_0^1 = \frac{37}{12}
\end{aligned}$$

挑戰範例

1. 求 $\lim_{x\to 0^+} x\cdot\int_x^1 \frac{\cos t}{t^2}\, dt = ?$ 《台大》

《提示》 ➥ $0\cdot\infty \Rightarrow \frac{\infty}{\infty}$

《解》 ☞ 因

$$\begin{aligned}\int_{0^+}^1 \frac{\cos t}{t^2}\, dt &= \int_{0^+}^1 \cos t\; d(-\frac{1}{t}) \\ &= -\frac{1}{t}\cos t\,\Big|_{0^+}^1 - \int_{0^+}^1 \frac{\sin t}{t} dt \\ &= \infty \quad (\text{因 } \frac{\sin t}{t} \text{ 於 } (0^+, 1) \text{ 爲有界 })\end{aligned}$$

故

$$\lim_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\, dt = \lim_{x\to 0^+}\frac{\int_x^1 \frac{\cos t}{t^2}\, dt}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{-\frac{\cos x}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\cos x = 1$$

2. Find a function f satisfying : $f(x) = 1 + \int_0^x \{1 + f(t)^2\}\cos t\; dt$. 《清大》

《解》 ☞ 因

$$f(x) = 1 + \int_0^x (1 + f(t)^2)\cos t\; dt \tag{1}$$

令 $y = f(x)$, 對 (1) 式兩端微分可得

$$\frac{dy}{dx} = \frac{d}{dx}f(x) = \frac{d}{dx}\Big[1 + \int_0^x \{1 + f(t)^2\}\cos t\; dt\Big] = (1 + y^2)\cos x$$

故

$$\int \frac{dy}{1+y^2} = \int \cos x \; dx$$

可得

$$\tan^{-1} y = \sin x + c$$

即

$$y = f(x) = \tan(\sin x + c)$$

又

$$f(0) = 1 + \int_0^0 \{1 + f(t)^2\} \cos t \; dt = 1$$

即

$$f(0) = \tan(\sin 0 + c) = 1$$

故 $c = \dfrac{\pi}{4}$，則

$$f(x) = \tan(\sin x + \frac{\pi}{4})$$

3. 假設

(a) $f(x)$ 為連續函數，且當 $x > 0$ 時，$f(x) > 0$。

(b) $x > 0$ 時，$\int_0^x f(t) \; dt = [f(x)]^2$

請求 (1) $f(0) = ?$ (2) $x > 0$ 時，$f(x) = ?$ 《台大商研所》

《解》 ☞

(1) 已知 $\int_0^x f(t) \; dt = [f(x)]^2$，故

$$f(x) = \sqrt{\int_0^x f(t) \; dt}$$

且 $f(x)$ 為連續函數，則

$$f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{\int_0^x f(t) \; dt} = 0$$

(2) $x > 0$ 時

$$\int_0^x f(t) \; dt = [f(x)]^2 \tag{1}$$

對 (1) 式兩端微分

$$\frac{d}{dx}[f(x)]^2 = \frac{d}{dx}\int_0^x f(t)\ dt$$

可得

$$2f(x)f'(x) = f(x)$$

因 $f(x) > 0$，故 $2f'(x) = 1$，則 $f'(x) = \dfrac{1}{2}$，因此

$$f(x) = \frac{x}{2} + c$$

又 $f(0) = 0 = c$，則 $c = 0$，故 $f(x) = \dfrac{x}{2}$。

4. 設 $f(0) = 1$，但 $t \neq 0$ 時 $f(t) = 1 + e^{-t^{-2}}$，令

$$F(x) = \int_{-1}^x f(t)\ dt$$

問 $F(x)$ 在 $x = 0$ 處是否可微分，請說明理由，若可微分，求 $F'(0)$。 《中央》

《解》 ☞ 因

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(1 + e^{-t^{-2}}) = \lim_{t\to 0}(1 + \frac{1}{e^{1/t^2}}) = 1 = f(0)$$

且 $t \neq 0$ 時，$f(t) = 1 + e^{-t^{-2}}$ 爲連續，故 $f(t)$ 於 $\mathbb{R}$ 上爲連續，則

$$\begin{aligned}
F'(0) &= \lim_{x\to 0}\frac{F(x) - F(0)}{x - 0} = \lim_{x\to 0}\frac{1}{x}\Big[\int_{-1}^x f(t)\ dt - \int_{-1}^0 f(t)\ dt\Big] \\
&= \lim_{x\to 0}\frac{1}{x}\Big[\int_{-1}^0 f(t)\ dt + \int_0^x f(t)\ dt - \int_{-1}^0 f(t)\ dt\Big] \\
&= \lim_{x\to 0}\frac{1}{x}\int_0^x f(t)\ dt \\
&= \lim_{x\to 0} f(x) = f(0) = 1
\end{aligned}$$

5. 求 $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\ dy$ 之值。 《中興》

《解》☞

$$\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \int_0^1 \frac{1}{1+y}\,d(y^n) = \frac{y^n}{1+y}\Big|_0^1 + \int_0^1 \frac{y^n}{(1+y)^2}\,dy$$
$$= \frac{1}{2} + \int_0^1 \frac{y^n}{(1+y)^2}\,dy \qquad (1)$$

因 y^n 為連續函數，故由積分均值定理可知，$\exists c \in (0\,,\,1)$，使得

$$\int_0^1 \frac{y^n}{(1+y)^2}\,dy = c^n \int_0^1 \frac{1}{(1+y)^2}\,dy = c^n\,(-\frac{1}{1+y})\,\Big|_0^1 = \frac{1}{2}c^n \qquad (2)$$

將 (2) 式代回 (1) 式可得

$$\lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \lim_{n\to\infty}(\,\frac{1}{2}+\frac{1}{2}c^n\,) = \frac{1}{2}$$

6. 求 $\displaystyle\int_0^1 \frac{x\,e^x}{(x+1)^2}\,dx$。

《解》☞

$$\int_0^1 \frac{x\,e^x}{(x+1)^2}\,dx = \int_0^1 \frac{x+1-1}{(x+1)^2}e^x\,dx$$
$$= \int_0^1 \left\{\frac{e^x}{x+1} - \frac{e^x}{(x+1)^2}\right\}dx$$
$$= \int_0^1 \frac{d}{dx}(\frac{e^x}{x+1})\,dx$$
$$= \frac{e^x}{x+1}\Big|_0^1 = \frac{e}{2} - 1$$

7. 試計算 $\displaystyle\int_0^{\frac{\pi}{2}} \sin^k\theta\,d\theta$ 與 $\displaystyle\int_0^{\frac{\pi}{2}} \cos^k\theta\,d\theta$ 之值。(k 為正整數)

《解》☞

(1) $k=1$, 故

$$\int_0^{\frac{\pi}{2}} \sin\theta \; d\theta = -\cos\theta \Big|_0^{\frac{\pi}{2}} = 1$$

$k=2$, 故

$$\int_0^{\frac{\pi}{2}} \sin^2\theta \; d\theta = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} \; d\theta = (\frac{\theta}{2} - \frac{\sin 2\theta}{4}) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$k \geq 3$, 故

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^k\theta \; d\theta &= \int_0^{\frac{\pi}{2}} \sin^{k-1}\theta d(-\cos\theta) \\
&= -\sin^{k-1}\theta \; \cos\theta \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos\theta \; d(\sin^{k-1}\theta) \\
&= \int_0^{\frac{\pi}{2}} \cos\theta \cdot (k-1)\sin^{k-2}\theta \; \cos\theta \; d\theta \\
&= (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta \; (1-\sin^2\theta) \; d\theta \\
&= (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta \; d\theta - (k-1)\int_0^{\frac{\pi}{2}} \sin^k\theta \; d\theta
\end{aligned}$$

將 $(k-1)\int_0^{\frac{\pi}{2}} \sin^k\theta \; d\theta$ 移項到等號左邊，可得

$$k\int_0^{\frac{\pi}{2}} \sin^k\theta \; d\theta = (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta \; d\theta$$

即

$$\int_0^{\frac{\pi}{2}} \sin^k\theta \; d\theta = \frac{k-1}{k}\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta \; d\theta$$

(a) 當 k 爲奇數且 $k \geq 3$ 時

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^k\theta \; d\theta &= \frac{k-1}{k}\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta \; d\theta \\
&= \frac{k-1}{k}\cdot\frac{k-3}{k-2}\int_0^{\frac{\pi}{2}} \sin^{k-4}\theta \; d\theta \\
&= \cdots \\
&= \frac{k-1}{k}\cdot\frac{k-3}{k-2}\cdot\frac{k-5}{k-4}\cdots\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin\theta \; d\theta \\
&= \frac{2\cdot 4\cdot 6\cdots(k-3)\cdot(k-1)}{3\cdot 5\cdot 7\cdots(k-2)\cdot k}
\end{aligned}$$

(b) 當 k 爲偶數且 $k \geq 3$ 時

$$\int_0^{\frac{\pi}{2}} \sin^k\theta \; d\theta$$

$$\begin{aligned}
&= \frac{k-1}{k}\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta\ d\theta \\
&= \frac{k-1}{k}\cdot\frac{k-3}{k-2}\int_0^{\frac{\pi}{2}} \sin^{k-4}\theta\ d\theta \\
&= \cdots \\
&= \frac{k-1}{k}\cdot\frac{k-3}{k-2}\cdot\frac{k-5}{k-4}\cdots\frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^{2}\theta\ d\theta \\
&= \frac{3\cdot5\cdot7\cdots(k-3)\cdot(k-1)}{4\cdot6\cdot8\cdots(k-2)\cdot k}\cdot\frac{\pi}{4}
\end{aligned}$$

(2) 令 $u=\frac{\pi}{2}-\theta$，則 $d\theta=-du$，且 $\begin{array}{c|c|c} \theta & 0 & \frac{\pi}{2} \\ \hline u & \frac{\pi}{2} & 0 \end{array}$，故

$$\int_0^{\frac{\pi}{2}} \cos^k\theta\ d\theta = \int_{\frac{\pi}{2}}^{0} \cos^k(\frac{\pi}{2}-u)(-du) = \int_0^{\frac{\pi}{2}} \sin^k u\ du = \int_0^{\frac{\pi}{2}} \sin^k\theta\ d\theta$$

則 $\int_0^{\frac{\pi}{2}} \cos^k\theta d\theta$ 之值與 (1) 之 $\int_0^{\frac{\pi}{2}} \sin^k\theta d\theta$ 值相同。

8. 求 $\lim\limits_{x\to0}\dfrac{\int_0^{\sin x}\sqrt{\tan x}\ dx}{\int_0^{\tan x}\sqrt{\sin x}\ dx}=?$

《提示》➥ $\frac{0}{0}$ 不定型。

《解》☞ 因 $\lim\limits_{x\to0}\int_0^{\sin x}\sqrt{\tan x}\ dx=0$ 且 $\lim\limits_{x\to0}\int_0^{\tan x}\sqrt{\sin x}\ dx=0$，故

$$\begin{aligned}
\lim_{x\to0}\frac{\int_0^{\sin x}\sqrt{\tan x}\ dx}{\int_0^{\tan x}\sqrt{\sin x}\ dx} &= \lim_{x\to0}\frac{\sqrt{\tan(\sin x)}\cdot(\cos x)}{\sqrt{\sin(\tan x)}\cdot\sec^2 x} \\
&= \lim_{x\to0}\left\{\frac{\tan(\sin x)}{\sin(\tan x)}\right\}^{1/2}\cdot\lim_{x\to0}\frac{\cos x}{\sec^2 x} \\
&= \left\{\lim_{x\to0}\frac{\tan(\sin x)}{\sin(\tan x)}\right\}^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \lim_{x\to 0} \frac{\sec^2(\sin x)\cdot\cos x}{\cos(\tan x)\cdot\sec^2 x} \right\}^{1/2} \\
&= 1
\end{aligned}
$$

9. 設函數 $f(x) = \int_0^x 3^t\ dt$，試利用微分均值定理證明：存在一數 c，$0 < c < 1$，使得 $\int_0^1 3^t\ dt = 3^c$。

《証》 ☞ 因 $f(x) = \int_0^x 3^t\ dt$ 由微分均值定理知，$\exists\, c \in (0\,,\,1)$，使得

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} \;\Rightarrow\; 3^c = f(1) - 0 = \int_0^1 3^t\ dt$$

故 $\int_0^1 3^t\ dt = 3^c$

10. 設 $f(x)$ 為連續函數，且 $\int_0^{x^2(1+x)} f(t)\ dt = x$，求 $f(2) = ?$ 《元智》

《解》 ☞ 因

$$\int_0^{x^2(1+x)} f(t)\ dt = x \tag{1}$$

對 (1) 式兩端微分可得

$$\frac{d}{dx}\int_0^{x^2(1+x)} f(t)\ dt = 1$$

即

$$f(x^2(1+x)) \cdot \{2x(1+x) + x^2\} = 1 \tag{2}$$

現令 $x^2(1+x) = 2$，即 $x^3 + x^2 - 2 = 0$，故

$$(x-1)(x^2+2x+2) = 0$$

可解得 $x = 1$，代回 (2) 式可得

$$f(2)\cdot 5 = 1$$

故 $f(2) = \dfrac{1}{5}$

11. 求下列積分

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x}\, dx$ (b) $\displaystyle\int_1^2 \frac{x^4+2x^2+2x+2}{x^3+2x}\, dx$ 《交大》

(c) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\, dx$ (d) $\displaystyle\int_{-1}^0 \sqrt{\frac{1+x}{1-x}}\, dx$

《解》 ☞

(a) 令 $z = \tan\dfrac{x}{2}$，故 $dx = \dfrac{2}{1+z^2}dz$、$\sin x = \dfrac{2z}{1+z^2}$、$\cos x = \dfrac{1-z^2}{1+z^2}$，且

x	0	$\frac{\pi}{2}$
z	0	1

故

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{1}{\sin x+\cos x}\, dx &= \int_0^1 \frac{1}{\dfrac{2z}{1+z^2}+\dfrac{1-z^2}{1+z^2}} \cdot \frac{2}{1+z^2}dz \\ &= \int_0^1 \frac{2}{2z+1-z^2}dz = \int_0^1 \frac{2}{2-(z-1)^2}dz \\ &= \frac{2}{\sqrt{2}}\tanh^{-1}\frac{z-1}{\sqrt{2}}\Big|_0^1 = -\frac{2}{\sqrt{2}}\tanh^{-1}\frac{-1}{\sqrt{2}}\end{aligned}$$

(b) 因

$$\frac{x^4+2x^2+2x+2}{x^3+2x} = x + \frac{2x+2}{x(x^2+2)}$$

令

$$\frac{2x+2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2} \tag{1}$$

故

$$A = \frac{2x+2}{(x^2+2)}\Big|_{x=0} = 1$$

(1) 式兩端乘 x 取 $x \to \infty$，可得

$$0 = A + B \;\Rightarrow\; B = -1$$

再令 $x=1$ 代回 (2) 式可得

$$\frac{4}{3}=A+\frac{B+C}{3} \ \Rightarrow \ C=2$$

故

$$\frac{x^4+2x^2+2x+2}{x^3+2x}=\frac{1}{x}+\frac{-x+2}{x^2+2}$$

則

$$\begin{aligned}
&\int_1^2 \frac{x^4+2x^2+2x+2}{x^3+2x}\ dx \\
=\ &\int_1^2 \{x+\frac{1}{x}+\frac{-x+2}{x^2+2}\}\ dx \\
=\ &\int_1^2 x\ dx+\int_1^2 \frac{1}{x}\ dx+\int_1^2 \frac{-x}{x^2+2}\ dx+2\int_1^2 \frac{1}{x^2+2}\ dx \\
=\ &\frac{x^2}{2}\ \Big|_1^2+\ln|x|\ \Big|_1^2-\frac{1}{2}\ln(x^2+2)\ \Big|_1^2+\frac{2}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\ \Big|_1^2 \\
=\ &2-\frac{1}{2}+\ln 2-\ln 1-\frac{1}{2}(\ln 6-\ln 3)+\frac{2}{\sqrt{2}}(\tan^{-1}\sqrt{2}-\tan^{-1}\frac{1}{\sqrt{2}}) \\
=\ &\frac{3}{2}+\ln\sqrt{2}+\sqrt{2}(\tan^{-1}\sqrt{2}-\tan^{-1}\frac{1}{\sqrt{2}})
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}dx &= \int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\ dx \\
&= \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{\cos^2 x}dx \\
&= \int_0^{\frac{\pi}{3}} \sec^2 x\, dx-\int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\ d\cos x \\
&= \tan x\ \Big|_0^{\frac{\pi}{3}}+\frac{1}{\cos x}\ \Big|_0^{\frac{\pi}{3}} \\
&= \sqrt{3}-0+(2-1) \\
&= \sqrt{3}+1
\end{aligned}$$

(d)

$$\begin{aligned}
\int_{-1}^0 \sqrt{\frac{1+x}{1-x}}\ dx &= \int_{-1}^0 \sqrt{\frac{1+x}{1-x}\cdot\frac{1+x}{1+x}}dx=\int_{-1}^0 \frac{1+x}{\sqrt{1-x^2}}dx \\
&= \int_{-1}^0 \frac{1}{\sqrt{1-x^2}}dx+\int_{-1}^0 \frac{x}{\sqrt{1-x^2}}dx \\
&= \sin^{-1}x\ \Big|_{-1}^0-\sqrt{1-x^2}\ \Big|_{-1}^0 \\
&= \sin^{-1}0-\sin^{-1}(-1)-(1-0) \\
&= \frac{\pi}{2}-1
\end{aligned}$$

12. 設 $f(x) = \int_{\pi}^{x} \sqrt{1+\cos^2 t}\ dt$，求 $(f^{-1})'(0)$。

《解》 ☞ 因 $f^{-1}(f(x)) = x$，兩邊對 x 微分可得

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

故

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

當 $x = \pi$ 時，

$$f(\pi) = \int_{\pi}^{\pi} \sqrt{1+\cos^2 t}\ dt = 0$$

則

$$(f^{-1})'(0) = \frac{1}{\left(\frac{d}{dx}\int_{\pi}^{x} \sqrt{1+\cos^2 t}\ dt\right)\Big|_{x=\pi}} = \frac{1}{\sqrt{1+\cos^2 x}\ \Big|_{x=\pi}} = \frac{1}{\sqrt{2}}$$

13. 求下列積分

(a) $\int_{-2}^{2} |x^3 - 1|\ dx$　　(b) $\int_{0}^{\frac{\pi}{2}} |\sin x - \cos x|\ dx$

《解》 ☞

(a) 令 $x^3 - 1 = 0$，則 $(x-1)(x^2+x+1) = 0$，故 $x = 1$

$$\begin{aligned}
\int_{-2}^{2} |x^3 - 1|\ dx &= \int_{-2}^{1} (-x^3 + 1)\ dx + \int_{1}^{2} (x^3 - 1)\ dx \\
&= \left(-\frac{x^4}{4} + x\right)\Big|_{-2}^{1} + \left(\frac{x^4}{4} - x\right)\Big|_{1}^{2} \\
&= -\frac{1}{4} + 1 - (-4 - 2) + 4 - 2 - \left(\frac{1}{4} - 1\right) \\
&= \frac{19}{2}
\end{aligned}$$

(b) 令 $\sin x - \cos x = 0$，則 $\dfrac{\sin x}{\cos x} = 1$，故 $x = \dfrac{\pi}{4}$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} |\sin x - \cos x|\ dx &= \int_0^{\frac{\pi}{4}} (-\sin x + \cos x)\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\ dx \\
&= (\cos x + \sin x)\ \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\ \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) - (1+0) + (0-1) - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) \\
&= 2\sqrt{2} - 2
\end{aligned}$$

14. 求下列積分

(a) $\displaystyle\int_{-1}^{2} [x]\ dx$

(b) $\displaystyle\int_{\frac{1}{4}}^{1} [\,\frac{1}{x}\,]\ dx$

(c) $\displaystyle\int_0^2 [2x]\ dx$

(d) $\displaystyle\int_0^2 [x^2]\ dx$

《解》 ☞

(a)

$$\begin{aligned}
\int_{-1}^{2} [x]\ dx &= \int_{-1}^{0} (-1) dx + \int_0^1 0\ dx + \int_1^2 1\ dx \\
&= -x\ \Big|_{-1}^{0} + 0 + x\ \Big|_1^2 \\
&= 0 - 1 + 2 - 1 = 0
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{\frac{1}{4}}^{1} [\,\frac{1}{x}\,]\ dx &= \int_{\frac{1}{4}}^{\frac{1}{3}} 3\ dx + \int_{\frac{1}{3}}^{\frac{1}{2}} 2\ dx + \int_{\frac{1}{2}}^{1} 1\ dx \\
&= 3x\ \Big|_{\frac{1}{4}}^{\frac{1}{3}} + 2x\ \Big|_{\frac{1}{3}}^{\frac{1}{2}} + x\ \Big|_{\frac{1}{2}}^{1} \\
&= 1 - \frac{3}{4} + 1 - \frac{2}{3} + 1 - \frac{1}{2} = \frac{13}{12}
\end{aligned}$$

(c)

$$\int_0^2 [2x]\ dx = \int_0^{\frac{1}{2}} 0\ dx + \int_{\frac{1}{2}}^{1} 1\ dx + \int_1^{\frac{3}{2}} 2\ dx + \int_{\frac{3}{2}}^{2} 3\ dx$$

$$= 0 + x\Big|_{\frac{1}{2}}^{1} + 2x\Big|_{1}^{\frac{3}{2}} + 3x\Big|_{\frac{3}{2}}^{2}$$
$$= 1 - \frac{1}{2} + 3 - 2 + 6 - \frac{9}{2} = 3$$

(d)

$$\begin{aligned}
\int_0^2 [x^2]\,dx &= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx \\
&= 0 + x\Big|_1^{\sqrt{2}} + 2x\Big|_{\sqrt{2}}^{\sqrt{3}} + 3x\Big|_{\sqrt{3}}^{2} \\
&= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} \\
&= 5 - \sqrt{2} - \sqrt{3}
\end{aligned}$$

15. 求 $\int_0^{\frac{\pi}{2}} \min\{\frac{3}{2\pi}x, \cos x\}\,dx$

《解》 ☞ 令 $\frac{3}{2\pi}x - \cos x = 0$，故 $x = \frac{\pi}{3}$，且 $0 < x < \frac{\pi}{3}$ 時，$\frac{3}{2\pi}x < \cos x$，則

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \min\{\frac{3}{2\pi}x, \cos x\}\,dx &= \int_0^{\frac{\pi}{3}} \frac{3}{2\pi}x\,dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos x\,dx \\
&= \frac{3}{4\pi}x^2\Big|_0^{\frac{\pi}{3}} + \sin x\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \frac{3}{4\pi}\cdot\frac{\pi^2}{9} - 0 + 1 - \frac{\sqrt{3}}{2} \\
&= \frac{\pi}{12} + 1 - \frac{\sqrt{3}}{2}
\end{aligned}$$

16. 求 $\int_{-1}^{2} x^2\,\mathrm{sgn}(x)\,dx$

《解》 ☞

$$\int_{-1}^{2} x^2\,\mathrm{sgn}(x)\,dx = \int_{-1}^{0}(-x^2)\,dx + \int_0^2 x^2\,dx$$

$$
\begin{aligned}
&= -\frac{x^3}{3}\Big|_{-1}^{0} + \frac{x^3}{3}\Big|_{0}^{2} \\
&= 0 - \frac{1}{3} + \frac{8}{3} - 0 = \frac{7}{3}
\end{aligned}
$$

17. 求 $\displaystyle\int_2^{17} \frac{dx}{\sqrt{\sqrt{x-1}+(x-1)^{5/4}}}$ 《淡大》

《解》 ☞ 令 $t=(x-1)^{\frac{1}{4}}$, 故 $x-1=t^4$, 則 $dx=4t^3\ dt$, 且

x	2	17
t	1	2

則

$$\int_2^{17} \frac{dx}{\sqrt{\sqrt{x-1}+(x-1)^{5/4}}} = \int_1^2 \frac{4t^3dt}{\sqrt{t^2+t^5}} = \int_1^2 \frac{4t^2}{\sqrt{1+t^3}}\ dt$$

再令 $1+t^3=u^2$, 故 $3t^2\ dt=2u\ du$, 且

t	1	2
u	$\sqrt{2}$	3

故

$$
\begin{aligned}
\int_2^{17} \frac{dx}{\sqrt{\sqrt{x-1}+(x-1)^{5/4}}} &= \int_1^2 \frac{4t^2}{\sqrt{1+t^3}}\ dt \\
&= \int_{\sqrt{2}}^{3} \frac{\frac{4}{3}\cdot 2udu}{u} \\
&= = \frac{8}{3}\int_{\sqrt{2}}^{3} du = \frac{8}{3}u\ \Big|_{\sqrt{2}}^{3} \\
&= \frac{8}{3}(3-\sqrt{2})
\end{aligned}
$$

18. (a) 試證 $\displaystyle\int_0^\infty \frac{x^c}{(1+x^c)(1+x^2)}\,dx = \int_0^\infty \frac{1}{(1+x^c)(1+x^2)}\,dx$

(b) 由 (a) 求 $\displaystyle\int_0^\infty \frac{x^c}{(1+x^c)(1+x^2)}\,dx$ 之值。

《解》 ☞

(a) 令 $x = \dfrac{1}{t}$，故 $dx = -\dfrac{1}{t^2}\,dt$，且

x	0	∞
t	∞	0

$$\begin{aligned}\int_0^\infty \frac{x^c}{(1+x^c)(1+x^2)}\,dx &= \int_\infty^0 \frac{\frac{1}{t^c}}{(1+\frac{1}{t^c})(1+\frac{1}{t^2})}\cdot(-\frac{1}{t^2})\,dt \\ &= \int_0^\infty \frac{1}{(t^c+1)(t^2+1)}\,dt \\ &= \int_0^\infty \frac{1}{(1+x^c)(1+x^2)}\,dx\end{aligned}$$

(b) 令

$$I = \int_0^\infty \frac{x^c}{(1+x^c)(1+x^2)}\,dx \quad (1)$$

$$I = \int_0^\infty \frac{1}{(1+x^c)(1+x^2)}\,dx \quad (2)$$

(1) 式 + (2) 式，可得

$$2I = \int_0^\infty \frac{1+x^c}{(1+x^c)(1+x^2)}dx = \int_0^\infty \frac{1}{1+x^2}dx = \tan^{-1}x\,\Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

故

$$\int_0^\infty \frac{x^c}{(1+x^c)(1+x^2)}dx = \frac{\pi}{4}$$

本題之 (b) 小題亦可直接令 $x = \tan\theta$ 解之。

19. 求 $\displaystyle\int_0^{2\pi} \frac{1}{1+\tan^4 x}dx$ 之值。

《解》 ☞

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{1+\tan^4 x}\ dx \\
= &\int_0^{2\pi} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx \\
= &\int_0^{\pi} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx + \int_{\pi}^{2\pi} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx \\
&(\text{ 令 } t = 2\pi - x\text{ , 故 } dx = -dt\ ) \\
= &\int_0^{\pi} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx + \int_{\pi}^{0} \frac{\cos^4(2\pi - t)}{\cos^4(2\pi - t)+\sin^4(2\pi - t)}\ (-dt) \\
= &\ 2\int_0^{\pi} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx \\
= &\ 2\Big[\int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x+\sin^4 x} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx\Big] \\
&(\text{ 令 } t = \pi - x\text{ , 故 } dx = -dt\ ) \\
= &\ 2\Big[\int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx + \int_{\frac{\pi}{2}}^{0} \frac{\cos^4(\pi - t)}{\cos^4(\pi - t)+\sin^4(\pi - t)}\ (-dt)\Big] \\
= &\ 4\int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx
\end{aligned}$$

設

$$I = 4\int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x+\sin^4 x}\ dx \tag{1}$$

令 $t = \dfrac{\pi}{2} - x$, 故 $dx = -dt$, 則

$$\begin{aligned}
I &= 4\int_{\frac{\pi}{2}}^{0} \frac{\cos^4(\frac{\pi}{2} - t)}{\cos^4(\frac{\pi}{2} - t)+\sin^4(\frac{\pi}{2} - t)}(-dt) \\
&= 4\int_0^{\frac{\pi}{2}} \frac{\sin^4 t}{\sin^4 t+\cos^4 t}dt = 4\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\cos^4 x+\sin^4 x}dx
\end{aligned} \tag{2}$$

(1) 式 + (2) 式 , 可得

$$2I = 4\int_0^{\frac{\pi}{2}} \frac{\cos^4 x+\sin^4 x}{\cos^4 x+\sin^4 x}\ dx = 4\int_0^{\frac{\pi}{2}}\ dx = 4x\ \Big|_0^{\frac{\pi}{2}} = 2\pi$$

故知

$$I = \int_0^{2\pi} \frac{1}{1+\tan^4 x}dx = \pi$$

20. 求 $\displaystyle\int_0^{\pi} \frac{x\sin x}{8+\sin^2 x}\, dx$

《解》 ☞ 令 $t=\pi-x$, 故 $dx=-dt$, 則

$$\begin{aligned}\int_0^{\pi} \frac{x\sin x}{8+\sin^2 x}\, dx &= \int_{\pi}^{0} \frac{(\pi-t)\sin(\pi-t)}{8+\sin^2(\pi-t)}(-dt) = \int_0^{\pi} \frac{(\pi-t)\sin t}{8+\sin^2 t}\, dt \\ &= \pi\int_0^{\pi} \frac{\sin t}{8+\sin^2 t}dt - \int_0^{\pi} \frac{t\sin t}{8+\sin^2 t}\, dt\end{aligned}$$

將 $\displaystyle\int_0^{\pi} \frac{t\sin t}{8+\sin^2 t}\, dt$ 移項到等號的左邊可得

$$2\int_0^{\pi} \frac{x\sin x}{8+\sin^2 x}\, dx = \pi\int_0^{\pi} \frac{\sin x}{8+\sin^2 x}\, dx$$

故

$$\begin{aligned}\int_0^{\pi} \frac{x\sin x}{8+\sin^2 x}\, dx &= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{8+\sin^2 x}\, dx \\ &\quad (\text{ 令 } u=\cos x \text{ , 則 } du=-\sin x\, dx \text{)} \\ &= \frac{\pi}{2}\int_1^{-1} \frac{1}{9-u^2}(-du) = \frac{\pi}{2}\int_{-1}^{1} \frac{1}{9-u^2}\, du \\ &= \pi\int_0^{1} \frac{1}{9-u^2}\, du = \frac{\pi}{6}\ln\left|\frac{u+3}{u-3}\right|\,\Bigg|_0^1 \\ &= \frac{\pi}{6}\ln 2\end{aligned}$$

21. 求 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x+\cos x}\, dx$

《解》 ☞ 設

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x+\cos x}\, dx \qquad (1)$$

令 $t = \frac{\pi}{2} - x$，則 $dx = -dt$，故

$$I = \int_{\frac{\pi}{2}}^{0} \frac{\sin^3(\frac{\pi}{2} - t)}{\sin(\frac{\pi}{2} - t) + \cos(\frac{\pi}{2} - t)}(-dt) \quad (2)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^3 t}{\cos t + \sin t}\, dt = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x + \cos x}\, dx \quad (3)$$

(1) 式 + (2) 式，可得

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x}\, dx \\ &= \int_0^{\frac{\pi}{2}} (\sin^2 x - \sin x \cos x + \cos^2 x) dx \\ &= \int_0^{\frac{\pi}{2}} (1 - \frac{1}{2}\sin 2x)\, dx \\ &= (x + \frac{1}{4}\cos 2x)\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} - \frac{1}{4} - (0 + \frac{1}{4}) \\ &= \frac{\pi}{2} - \frac{1}{2} \end{aligned}$$

故

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x} dx = \frac{\pi}{4} - \frac{1}{4}$$

22. 求 $\int_0^{\frac{2\pi}{3}} \frac{1}{5 + 4\cos\theta}\, d\theta$

《解》 ☞ 令 $z = \tan\frac{\theta}{2}$，$\cos\theta = \frac{1 - z^2}{1 + z^2}$，$d\theta = \frac{2}{1 + z^2}dz$

x	0	$\frac{2\pi}{3}$
z	0	$\sqrt{3}$

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5 + 4\cos\theta}\, d\theta = \int_0^{\sqrt{3}} \frac{\frac{2}{1 + z^2}}{5 + 4 \cdot \frac{1 - z^2}{1 + z^2}}\, dz = \int_0^{\sqrt{3}} \frac{2}{5(1 + z^2) + 4(1 - z^2)}\, dz$$

$$= \int_0^{\sqrt{3}} \frac{1}{9 + z^2}\, dz = \frac{1}{3}\tan^{-1}\frac{z}{3}\Big|_0^{\sqrt{3}} = \frac{\pi}{9}$$